MATH 451: Graph Theory

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Recall:

- $\kappa(G)$ = minimum number of vertices necessary to remove to make a graph either disconnected or trivial.
- $\lambda(G)$ = minimum number of edges required to make a graph disconnected.

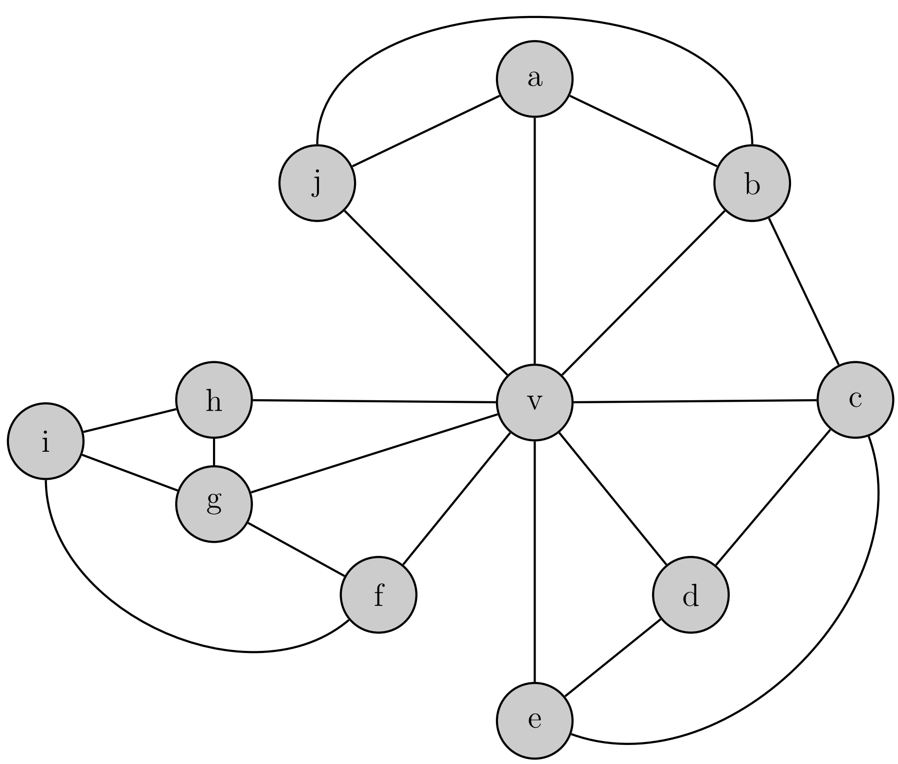
Example: Consider a country's transportation network.

1. Consider the "air traffic graph" where vertices are airports and edges are scheduled flights between them.
2. Also consider the highway graph where vertices are major cities and edges are the highways between them.

Which is the better measure of connectedness in each case?

In the first (1) case, $\kappa(G)$ is a better measure of connectedness.
For the second (2) case, $\lambda(G)$ is the better measure of connectedness.

Consider the graph G below:



$G - v$ is disconnected. In this graph, $\kappa(G) = 1$ and $\lambda(G) = 3$.

Note: $0 \leq \lambda(G) \leq n-1$. (You can isolate any vertex by removing all of the edges connected to it.)

In general, $\lambda(G) \leq \delta(G)$ for the same reason.

Theorem: $\lambda(K_n) = n-1$.

Proof. $\lambda(K_n) \leq \delta(K_n) = n-1$. We still need to show that $\lambda(K_n) \geq n-1$.

Aside: If U is a vertex cut of G that is minimal ($|U| = \kappa(G)$) then $G-U$ is disconnected. So, $G-U$ has components $G_1, G_2, \cdots, G_k$. Note that if $u \in U$ then u is adjacent to a vertex in each of $G_1, G_2, \cdots, G_k$. If X is a minimal edge cut then $G-X$ has exactly two components, G_1 and G_2, and every edge in X connects a vertex in G_1 to a vertex in G_2.

Proof Cont. Suppose X is a minimal edge cut of K_n. Since X is minimal, it divides K_n into 2 components, G_1 and G_2. Let's suppose: $|G_1| = j$ and $|G_2| = n-j$. Then, X contains every edge between a vertex of G_1 and a vertex of G_2. Then X contains an edge corresponding to every pair of vertices in G_1/G_2. So, $|X| = j(n-j) \geq n-1$. □

Theorem: $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Proof. $\lambda(G) \leq \delta(G)$ $\checkmark$

It remains to show that $\kappa(G) \leq \lambda(G)$. To do this, we break the proof into cases:

Case $1 : G = K_n$.

If $G = K_n$ then $\kappa(G) = n-1$ and $\lambda(G) = n-1$ $\checkmark$

Case $2 : G$ is disconnected.

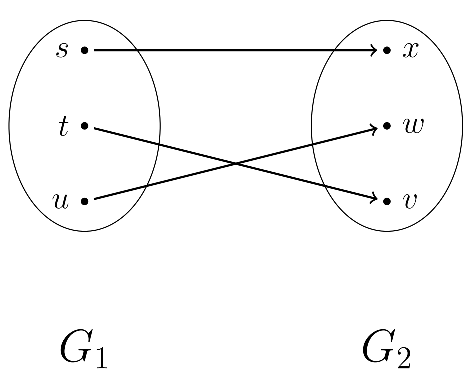
$\lambda(G) = \kappa(G) = 0$ $\checkmark$

Now we can assume G is a connected graph that is not the complete graph. Let's suppose G has order n and X is a minimum edge cut. So, $|X| = \lambda(G)$. Furthermore, $G - X$ has 2 components, G_1 and G_2, such that $|G_1| = j$ and $|G_2| = n-j$.

Case $3 :$ Suppose X contains every edge between G_1 and G_2.

Then, $|X| = j(n-j)$ (by the last proof). Since this minimized when $j = 1$, we get $\lambda(G) = n-1$. Since $\kappa(G) \leq n-1$ we get $\kappa(G) \leq \lambda(G)$.

Now we can assume that X does not connect every vertex of G_1 to every vertex of G_2. Let $u \in G_1$ and $v \in G_2$ be vertices such that $uv \notin G$. Consider the diagram below:



We use this to construct a vertex cut U. For every edge in X, if u is one vertex of e ($e = uw$ in this case) we put the vertex on the other end of e (the vertex in G_2) and put that in U. Otherwise, if u is not one of the vertices of e, we put the vertex in the G_1 side of e in U. U contains one of the vertices on either end of every edge in X. $|U| \leq |X|$ and $G - U$ contains none of the edges in X. Note that u,$v \notin U$. So, G_1 and G_2 didn't get completely removed in $G - U$. Since $G - X$ was disconnected, there is no longer a path from u to v. So, $\kappa(G) \leq |U| \leq |X| = \lambda(G)$. $\square$