Math 451 Notes

Phil Switlick

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Recall from last class

Non Seperable Graph

A non seperable graph is a graph without any cut vertices.

Theorem

A connected graph is non separable if and only if every two distinct vertices lie on a common cycle. Proved last class.

Blocks

Definition

A block in a graph is a maximal non separable subgraph. (In other words a non separable subgraph that is not contained in any other non separable subgraph)

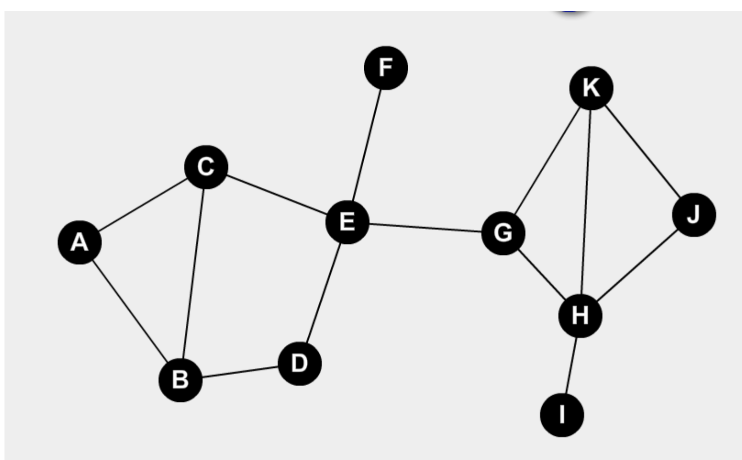

Figure 1: Blocks: {A,B,C,D,E}, {G,H,J,K}, {E,F}, {E,G}, {H,I}

Note

A block will always be an induced subgraph. (All the edges between vertices of the block are included.)

Theorem

Let $\sim$ be a relation on the edges of a connected graph G where for e, f $\in$ E(G), we say e $\sim$ f if

1. e = f

2. e and f lie on a common sycle of G

then $\sim$ is an equivalence relation.

Proof

- Reflexive ✓
- Symmetric - If e and f are any two cyces and e $\sim$ then true by definition.
- Transitive - We need to show that if e $\sim$ f and f $\sim$ g then e $\sim$ g.

Proof

Since e and f lie on a common cycle C, if g is part of this cycle then we are done. Otherwise since f $\sim$ g, f and g both lie on some other cycle C′. Suppose e = uv.

- Let P be the path from u to v not containing e
- Let x be the first vertex on P that is also part of C′
- Let y be the last vertex that P and C′ have in common
- Let P′ be the other path from x to y not part of C

Now that g must be a part of P′ since G is not part of C. We get a cycle containing e and g by starting at u, following P to x then following P′ to y (going through g all the way). Finally returning to v along the remaining part of p.

Note

Using the equivalence relation on the edges of a graph, the blocks of a graph are the edge induced sub graphs of an equivalence class with respect to $\sim$

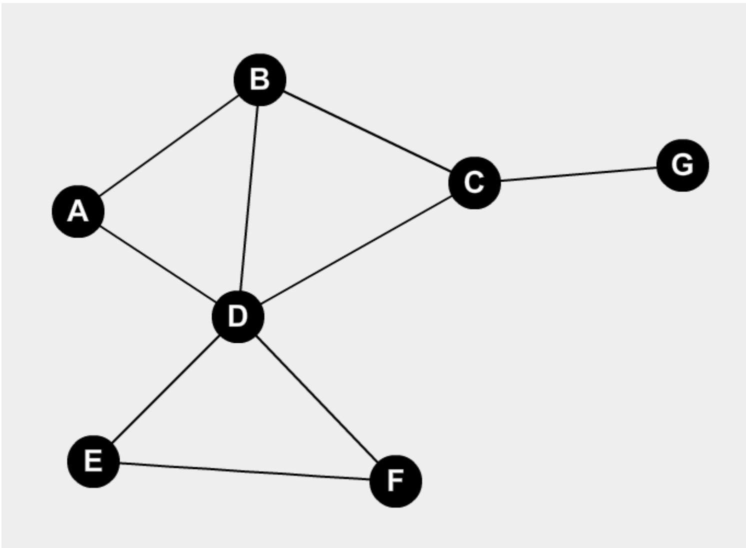

Figure 2: {AB $\sim$ BC $\sim$ CD $\sim$ DA $\sim$ BD}, {ED $\sim$ DF $\sim$ FE}

Corollary

If B1 and B2 are distinct blocks of G then

1. B1 and B2 have no edges in common

Proof

Follows immediately from the characterization as equivalence classes of edges.

2. B1 and B2 have at most one vertex in common

   Proof

   Proof by contradiction: Suppose B1 and B2 both contain distinct vertices u and v. That means there is a path p from u to v in B1, Also there is a path from v to u in B2. Using both of these paths we get a cycle in the original graph containing edges from both B1 and B2. This is a contradiction with B1 and B2 distinct blocks

3. If v $\in$ B1 and v $\in$ B2 then v is a cut vertex

   Proof

   Suppose that v is in B1 and B2. Suppose for contradiction that v is not a cut vertex. v is adjacent to some vertex v1 in B1 and some vertex v2 in B2. By our characterization of cut vertices there must be a path from v1 to v2 not going through v, call this path p. Then p along with v2,v and v,v1 form a cycle which contradicts B1 and B2 being separate blocks.

Vertex Cut

Definition

U $\subset$ V(G) is a vertex cut if G - U is disconnected

Minimum Vertex Cut

A vertex cut is a minimum vertex cut if no smaller set of vertices is a vertex cut.

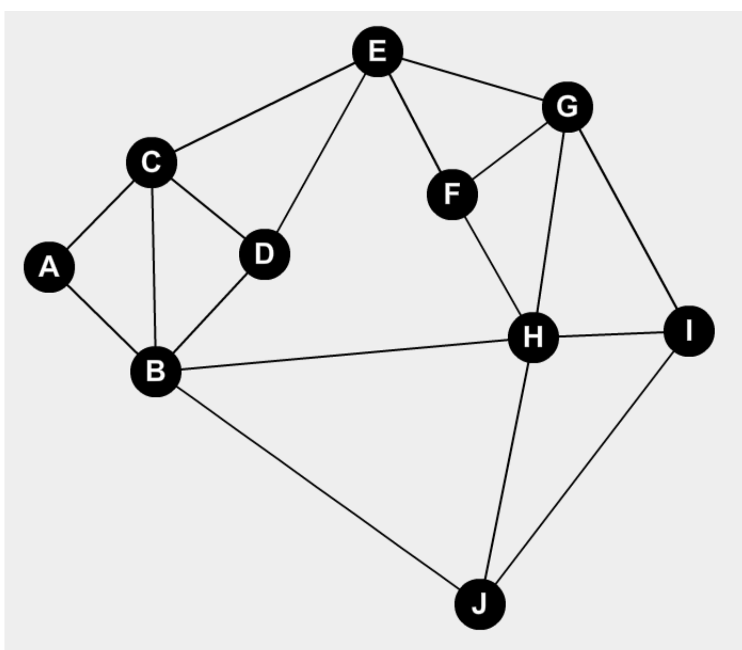

Figure 3: Minimum vertex cut: {B,E}, Another vertex cut: {A,D,F,G}

Connectivity

Define the connectivity of a graph G to be the size of a minimum cut vertex or the number of vertices that can be removed before the graph is trivial. Denote this as K(G)

- If a connected graph G has a cut vertex, $K(G) = 1$
- If graph G is disconnected, $K(G) = 0$
- If graph G is complete with order n, $K(Kn)$ = n - 1
- For any graph G with order n, $0 \leq K(G) \leq$ n - 1

Edge Cut

Definition

F $\subset$ E(G) is an edge cut if G - F is disconnected. An edge cut in Figure 3 is {CE,CD,BH,BJ}.

Minimum Edge Cut

Minimum edge cut is an edge cut such that no edge cut of G has a smaller size. Figure 3's minimum edge cut is {AB,AC}.

Edge Connectivity

The edge connectivity of a graph G is the size of a minimum edge cut. Denote this as Λ(G).