Graph Theory Day Four

February 8, 2018

1 Connected

Recall from last class, we discussed methods for proving a graph was connected. Our two methods were

1) Based on the definition, given any u, v $\in$ V(G), there exists a walk from u to v, then G is connected.
2) If G is a graph of order at least 3 with two vertices u,v $\in$ V(G) such that G - u and G - v are connected, then G is connected.

We will now use the degree of the vertices to determine whether a graph is connected.

If a graph of order n has a vertex of degree n-1, then the graph is connected.

Note this vertex must be connected to every other vertex since there are n-1 vertices available to be connected. As such, the graph must be connected.

The second rule we can use is a theorem that shall be used for many of our other stipulations.

Theorem If G is a graph of order n, and for every pair of nonadjacent vertices, u,v $\in$ V(G), we have deg u + deg v $\geq$ n - 1, then G is connected and the diameter is at most 2.

Proof Given G is a graph of order n with x, y $\in$ V(G) where deg x + deg y $\geq$ n-1.

Case 1: x, y are adjacent. Then x, y are connected and d(x, y) = 1 $\leq$ 2.

Case 2: x, y are not adjacent. Assume x,y do not share a common vertex, then the sum of their degrees would be at most n - 2 since there are n-2 other vertices in the graph. Thus, there must exists w such that xw, yw $\in$ E(G). As such, the walk xwy is a walk in G and therefore G is connected and the distance is equal to 2. ■

We can follow this theorem up with a corollary, a conclusion based on the previous theorem.

Corollary If G is a graph with $\delta(G) \geq \frac{n-1}{2}$, then G is connected.

Proof If u, v $\in$ G, then deg(u) + deg(v) $\geq \frac{n-1}{2} + \frac{n-1}{2} = n-1$. Therefore, G is connected by the previous theorem. ■

This is a great inequality, but can we go even steeper with our inequality? This is the concept of sharpness.

Sharp When we try to make the inequality any stronger in a mathematical result, the result is no longer true!

Is the corollary sharp? If we have a graph G such that $\delta(G) \geq \frac{n-2}{2}$, can we conclude for certain that G is connected?

Suppose n is even (n = 2k for k $\in$ **Z**). Create G with two components that are both the complete graph K_k. Then $\delta(G) = k-1 = \frac{2k-2}{2} = \frac{n-2}{2}$. But our graph is disconnected. As such, the theorem would not hold true for $\frac{n-2}{2}$.

2 Regular Graph

We transition to a new graph type: regular graphs.

A graph is considered **regular** if all vertices have the same degree. We write this as an **r-regular** graph if all vertices have degree r.

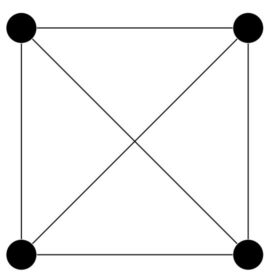
Examples
P_2 is 1-regular. P_K is not regular for $k \geq 3$ as the "endpoints" will have degree 1 but all points "in between" will have degree 2.

C_n is 2-regular for any $n \geq 3$.

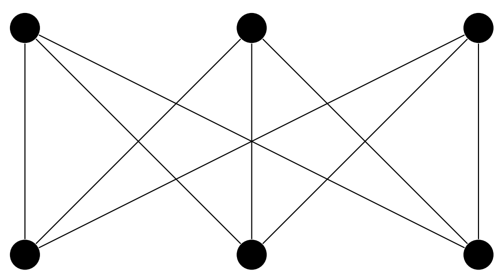
K_n is (n-1) - regular for any n.

2.1 Cubic Graph

A 3-regular graph is called a cubic graph. Some well known examples



K_4



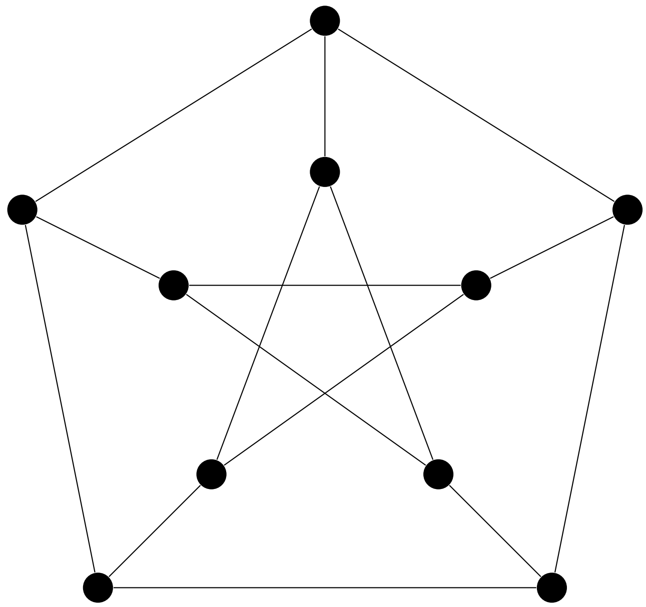
$K_{3,3}$

The notation of our second example is worth mentioning.

$K_{n,m}$: The complete bipartite graph. Given set U with n(U) = n and set V with n(V) = m where U, V $\in$ V(G), then each vertex in U is connected to each vertex in V.

Notice in $K_{3,3}$ the three vertices of the bottom row are connected to each of the vertices in the top row and vice versa.

One final poignant example to be mentioned in the future is the **Peterson Graph**.



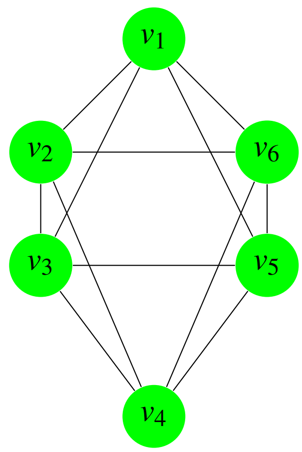
2.2 Possibility of r-regular Graphs

When does there exist an r-regular graph on n-vertices? Obviously, r $\leq$ n-1 since n-1 is the largest degree we can have at any point. Secondly, we know that we cannot have an odd number of odd degrees. As such, we need to look those situations where either r, n are both even or one of them is odd.

Theorem If $0 \leq r \leq n-1$ and n, r are not both odd, then there exists an r - regular graph on n-vertices.

Proof Case 1: Suppose r is even (r = 2k). Create the graph with n-vertices $(v_1, v_2, ..., v_n)$ Place these vertices around a circle. Connect each vertex to the k-vertices before and the k-vertices after it around the circle. Every vertex ends up being connected to exactly 2k vertices.

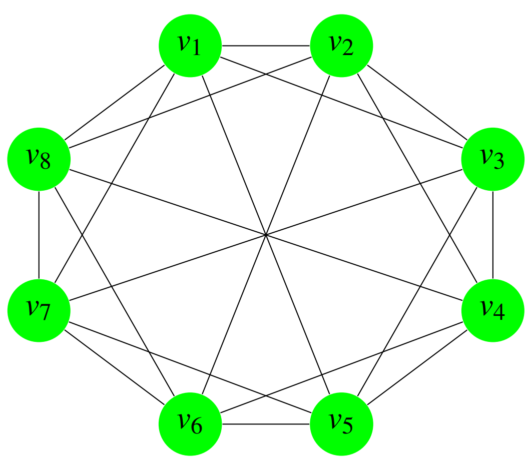
Example: 4-regular graph of order 6.



In this example, each vertex is connected to two vertices to the left and two vertices to the right since 4 = 2 (**2**).

Case 2: Assume r is odd (r = 2k + 1). As such, n must be even. Draw vertices around a circle. Connect the vertices to the k vertices before and after it. Since n is even, every vertex has an opposite vertex on the other side of the circle. Add the edges connecting opposite vertices as well. Thus, every vertex has degree 2k + 1.

Example: 5-regular graph of order 8



In this example, each vertex is connected to two vertices to the left and two vertices to the right since 5 = 2(**2**) + 1. The fifth vertex for the initial four vertices is found by adding four to the subscript of the vertex. This would represent the vertex across from it in the circle. Notice after the first four are made, the formula is not needed for the other half of the circle as these edges were already drawn.

Our next theorem looks at r-regular graphs and their subgraphs.

Theorem If G is any graph and r $\geq \Delta(G)$, then there exists an r-regular graph H such that G $\subseteq$ H. Furthermore, G is an induced subgraph of H. (Proof and construction of H found on pages 40-41.)

3 Degree Sequence

The next topic concerning degrees involves degree sequences.

A **degree sequence** $d_1, d_2, ..., d_n$ is a sequence of numbers in which all numbers are such that $0 \leq d_i \leq n-1$. This degree sequence is **graphical** if there exists a graph whose vertices have degree $d_1, d_2, ..., d_n$.

3.1 Examples

Is 1, 1, 1, 2 graphical? No. Two ways could be considered for this situation, both reflecting the common issue. 1) There are three odd degrees in this sequence. 2) The sum of the elements of the sequence is odd.

Is 1, 3, 3, 3 graphical? No. Since there are four vertices, a vertex with degree 3 represents a vertex which is connected to all other vertices. As such, there are three vertices which are connected to all other vertices. Therefore, a vertex of degree 1 is impossible.

Is 1, 2, 2, 3 graphical? We can make a graph using these vertices. As such, the sequence is graphical.