MATH 451 - Class Notes

2/6/2017

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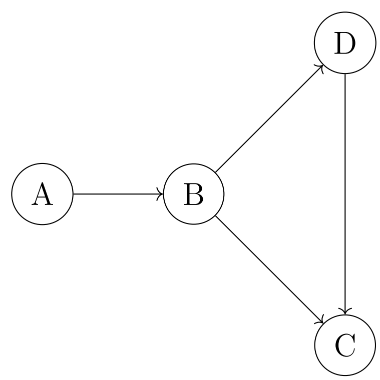
Summary:

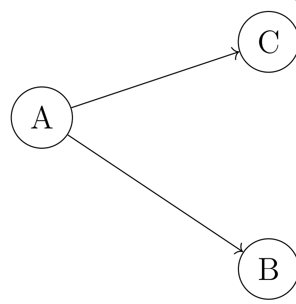
- Complement of a Graph
- Diameter of a Graph
- Bipartite Graphs

Def: The **complement** of a graph G denoted $\bar{G}$ is the graph with the same vertices as G but every edge of $\bar{G}$ is not in G and vice-versa.

The Graph G =



The Graph $\bar{G}$ =



AKA $\bar{G}$ has the same vertices as G but every edge in G is not an edge in $\bar{G}$ ad every edge in $\bar{G}$ is not in G.

Theorem: If G is disconnected then $\bar{G}$ is connected.
Proof: Suppose G is disconnected. Let X and Y be any two vertices of G (or $\bar{G}$).
Case 1: Suppose that X and Y are not adjacent in G. Then XY is an element of E(G) . So, X and Y are connected in $\bar{G}$.
Case 2: Suppose that X and Y are adjacent in G. Since G is disconnected, there must be some other vertex V in V(G) such that x is not connected to V and y is not connected to V. Since UV is not an element in E(G) and XV is an element in E(G) and likewise VY is not an element in E(G) and VY is an element in E(G). SO, (X,V,Y) is a walk in $\bar{G}$. Therefore, X and Y are connected in $\bar{G}$.

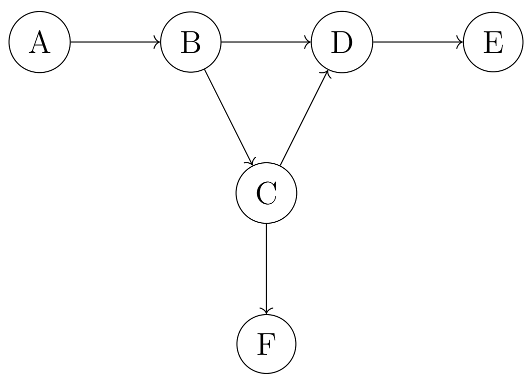
Last class we proved...

<u>Theorem:</u> If G is a graph of order at least 3 with 2 vertices U,V such that G-U and G-V are both connected then G is connected

Now we want to prove:
<u>Theorem:</u> If G is a connected graph of order at least 3 then there exists 2 vertices U and V such that G-U and G-V are both connected.

<u>Proof:</u> Take any two vertices in G such that the distance between them is equal to the diameter of G. Lets call these vertices U and V.
<u>Claim:</u> G-U and G-V are both connected.
Lets show that G-U is not connected. Then, there exists two vertices where there exists no path between them. Suppose one of these vertices is V, the other we call W. Since V and W were connected in G, there was a path from V to W in G that went through the vertex U. Every path from V to W in G passes through the vertex U. So, the distance from V to W is greater than the distance from V to U. This is a contradiction with the distance from U to V being the diameter of G.

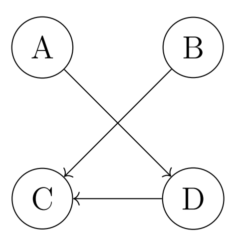
Def: the <u>diameter</u> of a graph G is the greatest distance between any two vertices X and Y in the graph G
Let the graph below be graph G:



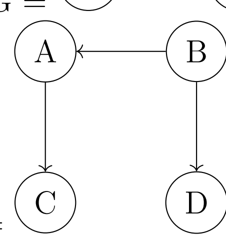
Ex.1 The diameter of vertices A and E is 3 because AB,BD,DE = distance of 3.
Ex.2 The diameter of vertices A and F is 3 because AB,DC,CF = distance of 3

<u>Theorem:</u> If G has order at least 3 then: G is connected iff there exist U,V is an element in V(G) AND G is connected iff G-U and G-V are connected, and vise-versa
Does there exist a graph G such that G and $\bar{G}$ are both connected?

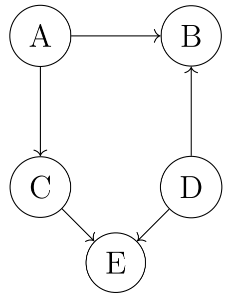
Ex 1. Graph G =



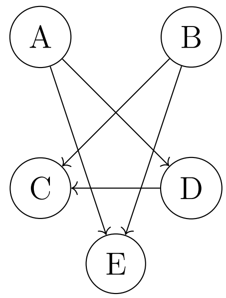
Graph $\bar{G}$ =



Note: Both of the graphs above are P_4 and we can say that yes, there exists a graph such that G and $\bar{G}$ are both connected.

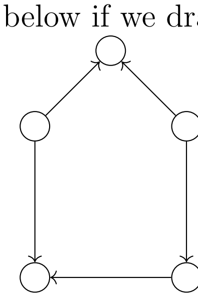


Ex 2. Graph G =



The graph $\bar{G}$ =

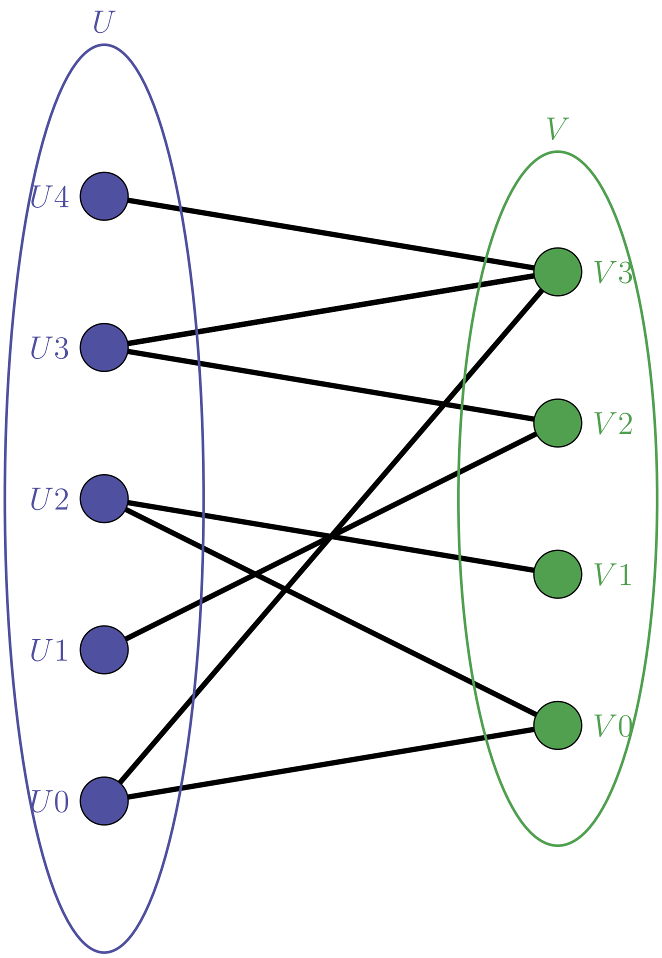
The image above is equivalent to the image below if we drag the vertices around to get rid of crossings.



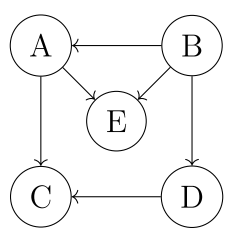
Note: Both of the graphs above are C_5 and there exists a graph such that G and $\bar{G}$ are both connected.

A graph G is called <u>bipartite</u> if the vertices of G can be partitioned into two sets U and V such that every edge of G connects a vertex of U with a vertex of V.

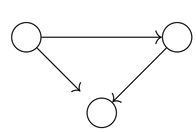
Ex. 1 (As you can see below, the vertices of G are partitioned into two sets U and V where every edge of G connects a vertex of U with a vertex of V.)



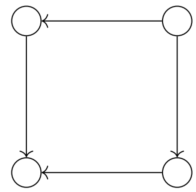
Ex. 2 - This graph can be considered a bipartite graph although we did not intend to create a bipartite graph. We can consider vertices C and B as part of set U and vertices A, E, and D a partition of set V.



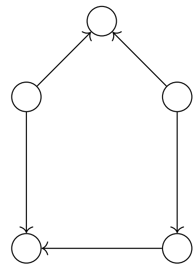
Ex. 3 - Is this bipartite? -NO, because it has a 3-cycle.



Ex. 3 - Is this bipartite? -YES, take opposite corners.



Ex. 4 - Is this bipartite? -NO, not possible.



C_n is a bipartite iff n is even.
Note: if any graph G contains an odd cycle as the subgraph,then it can't be bipartite.
<u>Theorem:</u> A graph G is bipartite iff it has no odd cycles as subgraphs
Proof: (You must show forward and reverse proof when iff appears in theorem)

<u>Forward Proof:</u> First we want to show that if G is bipartite, it has no odd cycles. If G has an odd cycle, then there s no way to partition the vertices around that cycle into U and V so it is not bipartite.

<u>Reverse Proof:</u> Suppose G has no odd cycles. We want to show G is bipartite. Suffices to prove that any connected components of G is bipartite. Suppose G is connected and has no odd cycles. Lets pick the element U in V(G). Let $U = \{V isanelementofV(G) : d(U,V)iseven\}$ and $V = \{V isanelementofV(G) : d(U,V)isodd\}$
<u>Claim:</u> the partition into U and V makes G a bipartite graph.
Need to show that there are no edges between vertices of U (or between vertices of V). Let's prove this for U. Suppose there exists X,Y element in U. So that, XY is an element in E(G). Since d(X,U)=2s is even, there exists a path $[U, V_1, V_2, ..., V_{2s-1}, X]$ from X to U. d(Y,U)=2t. There exists a path $[U, W_1, W_2, ..., W_{2t-1}, Y]$. Suppose that $[V_i]$ is the last vertex that appears in both of these paths. (Note: that $[V_i = W_i]$, the index where it appears in both paths are the same). Let's construct a cycle: $(V_i, V_{i+1}, V_{i+2}, ..., X, Y, W_{2t-1}, W_{2t-2}, ..., W_i)$. It is a cycle of length 2s+2t-i-i+1. So, (2s+2t-i-i+1)= odd cycle because of the plus 1. Then G contains an odd cycle. Note: the proof for W is the same as above and can also be found in the book.