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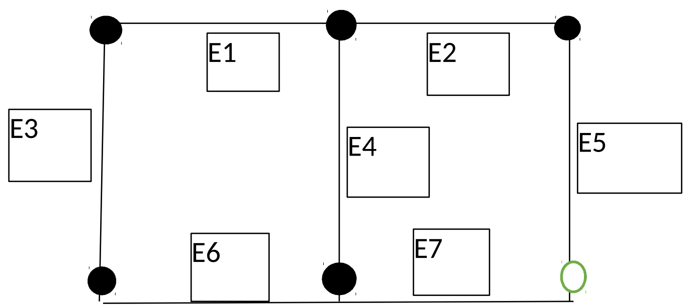
MATH 451

Dr. McNew

Lesson 4.4 Notes

- A **spanning tree** of G is a spanning subgraph of H of a connected graph G such that H is a tree.
- Example: Determine the number of spanning trees of the graph:

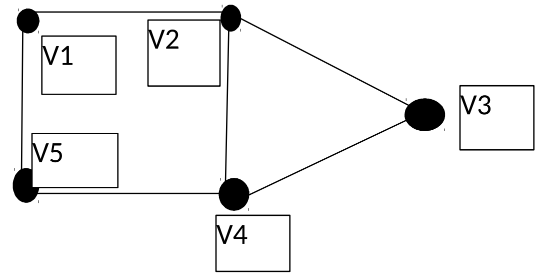


- Solution: Every spanning tree of G cannot have every edge of each cycle. We make take note of the number of spanning trees that have E4 and those that do not. First any spanning tree that does not have E4 must have five of the following: E1, E2, E3, E5, E6, and E7. Therefore, there are 6 spanning trees that do not contain E4. Then, any spanning tree that contains E4 must not contain exactly one of E1, E3, and E6 and must not contain exactly one of E2, E5, and E7. Therefore, there are 3 * 3 = 9 spanning trees that contain E4. Therefore, there are 6 + 9 = 15 spanning trees of G.
- The **Cayley Tree Formula** is the formula that computes the number of spanning trees of the graph $G = K_n$, where $V(G) = \{v_1, v_2, ..., v_n\}$, which is equivalent to the number of distinct trees with vertex set $\{v_1, v_2, ..., v_n\}$.
- Arthur Cayley established this formula in 1889.
- Theorem 4.15: The number of distinct trees of order n with a specified vertex set is n^{n-2}.
- Proof: First note that the statement is true for n = 1 (just one graph: K_1) and for n = 2 (just one graph: K_2). For n ≥ 3 we will give a bijection between the set T_S of all trees having vertex set S (with |S| = n) and the set S^{n-2} of sequences of length n−2. (We use a general n-element set S instead of specifying {1, 2, . . . , n} because we have to apply an induction assumption to different subsets of our starting set of vertices.)
- For |S| ≥ 3 define the function f_S recursively by $f_S(T) = (a_1, f_{S\setminus\{b1\}}(T_0))$, where b1 is the leaf of T with minimum label, a_1 is the label on the unique vertex of T adjacent to b1, and T_0 = T − b1. The recursion starts with $f_S(T)$ the empty string if |S| = 2. We show that fS is a bijection. Note that for |S| = 3, the only isomorphism type of tree is the path P3. There

are three different labeled graphs P3 with vertices {1, 2, 3}, and the function f_S from T_S to S is given by the middle vertex of the path, which clearly gives a bijection.

- Claim: The number of times i occurs in $f_S(T)$ is $\deg_T(i) - 1$. Why? The vertex label i occurs in $f_S(T)$ once for each time one of its neighbors is the minimum labeled leaf in the remaining tree. Before the recursion ends, $\deg_T(i) - 1$ of the neighbors of vertex i must have achieved the status of minimum labeled leaf and have been removed; each time i is appended to the sequence $f_S(T)$. After these $\deg_T(i) - 1$ neighbors have been removed, i is a leaf. The label i cannot be added to the sequence again. It could only be added to the sequence if it were the neighbor of another leaf, but that could only happen if the remaining tree were K_2, whose associated sequence is empty.
- Now we show that fS is one-to-one, by induction. We have already observed the $|S| = 3$ case. Assume $n \geq 4$ and for any two different trees with vertex set $|S_0| < n$, $f_{S0}(T_1) \neq f_{S0}(T_2)$. Let $|S| = n$, and let T_1 and T_2, $T_1 \neq T_2$, be trees with vertex set S. We will show that $f_S(T_1) \neq f_S(T_2)$.
- Case 1) The lowest labeled leaf c_1 of T_2 is different from the lowest labeled leaf b_1 of T_1. Without loss of generality $b_1 < c_1$. Then b_1 is not a leaf of T_2 (by minimality) of c_1, so $\deg T_2 (b_1) > 1$, so b_1 occurs in $f_S(T_2)$, but not in $f_S(T_1)$. So $f_S(T_1) \neq f_S(T_2)$.
- Case 2) The lowest labeled leaf of T_2 is the same as the lowest labeled leaf b_1 of T_1, but the neighbor of b_1 in T_2 is different from the neighbor of b_1 in T_1. Then the first entry of fS(T2) is different from the first entry in $f_S(T_1)$. So $f_S(T_1) \neq f_S(T_2)$.
- Case 3) The lowest labeled leaf of T_2 is the same as the lowest labeled leaf b_1 of T_1, and the neighbor of b_1 in T_2 is the same as the neighbor a_1 of b_1 in T_1, but $T_1 - b_1 \neq T_2 - b_1$. Then $f_S(T_1) = (a_1, f_{S\setminus\{b1\}}(T_1 - b_1))$ and $f_S(T_2) = (a_1, f_{S\setminus\{b1\}}(T_2 - b_1))$. By the induction assumption, $f_{S\setminus\{b1\}}(T_1 - b_1) \neq f_{S\setminus\{b1\}}(T_2 - b_1)$. So $f_S(T_1) \neq f_S(T_2)$.
- Thus f_S is one-to-one.
- It remains to show for $|S| \geq 3$ the function $f_S : T_S \to S^{n-2}$ is onto. Again, we use induction on n. For $|S| = 3$, the function is onto (see above). Assume $n \geq 4$ and for any $|S| = m < n$ and any sequence $a = (a_1, a_2, \dots, a_{m-2}) \in S^{m-2}$, there exists a tree T with vertex set S and $f_S(T) = a$. Now let $|S| = n$ and $a = (a_1, a_2, \dots, a_{n-2}) \in S^{n-2}$. Let b_1 be the least element of S not appearing in $\{a_1, a_2, \dots, a_{n-2}\}$, and $a_0 = (a_2, a_3, \dots, a_{n-2}) \in S^{n-3}$. Thus $a_0 \in (S \setminus \{b1\})^{n-3}$, so by induction there is a tree T_0 with vertex set $S \setminus \{b_1\}$ such that $f_{S\setminus\{b1\}}(T_0) = a_0$. Let T be the tree with vertex set S and $E(T) = E(T_0) \cup \{b_1a_1\}$. Then b_1 is a leaf of T, and it is the leaf with lowest label (since b_1 is the least element of S not appearing in $\{a_1, a_2, \dots, a_{n-2}\}$). So $f_S(T) = (a_1, f_{S\setminus\{b1\}}(T_0)) = (a_1, a_0) = (a_1, a_2, \dots, a_{n-2})$.
- Thus f_S is onto.
- So $f_S : T_S \to S^{n-2}$ is a bijection, and $|T_S| = |S^{n-2}| = n^{n-2}$.
- Example of Theorem 4.15: Let n = 3. There are 3 distinct trees of order 3 with a specified vertex set.
- **Theorem 4.16:** (Matrix Tree Theorem) Let G be a graph with $V(G) = \{v_1,v_2,\dots,v_n\}$, let $A = [a_{ij}]$ be the adjacency matrix of G and let $C = [c_{ij}]$ be the n x n matrix, where c_{ij}={deg v_i if i = j, and $-a_{ij}$ if $i \neq j$} Then the number of spanning trees of G is the value of any cofactor C.

- **Proof**: Since L = MM_0 , we have $L_0 = M_0M^t_0$. By Cauchy-Binet formula, we have $\det L_0$ = X S∈ X S∈([q] p−1) ($\det M_0[S]$)($\det M^t_0$ [S]) = X S $(\det(M_0[S]))^2$ (∵ $(A[S])^t = A^t[S]$) and $(\det(M0[S]))^2$ is 1 if S(E) forms a spanning tree and 0 otherwise, and a tree with p vertices has p − 1 edges, so summing over S ∈, RHS exactly counts the number of spanning trees of G.
- **Example**: Let G:



- A = F $\alpha\beta=$ $\begin{matrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 1 & 0 & 0 & -1 & 2 \end{matrix}$
- Then we solve by using the determinant of the matrix $\begin{matrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{matrix}$.
- The number of spanning trees is 2*(3*2-(-1*-1))+1(-1*2)=8.