Class Notes

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Remember from last time:

Theorem: G is a tree if and only if every edge of G is a bridge.

Claim: If G is a tree of order n then G has size $n-1$.

Definitions:

1. A tree is a connected graph with no cycles.
2. A bridge is a edge which when removed, disconnects the graph

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Claim: If G is a tree of order n then G has size $n-1$.

Proof. (Want to use Induction)
(Decide base case, induction case, and variable to perform induction on)

(Base case) See that if $n = 1$, the graph G has order one and a size of $n-1$, meaning it is the trivial tree with 0 edges.
Can also see that if $n = 2$, then the resulting tree has two vertices and one edge and follows the pattern that the size is $n-1$.

(Induction Step)
Assume the theorem is true for all trees of order n.
(Need to prove theorem is true for the next order of trees or $n+1$)
Let T be a tree with $n+1$ vertices.
(Goal: Prove T has n edges)
If this is true it cannot have order $n = 1$, it must have order at least two and thus is not the trivial tree.
(We proved last class that if T is a nontrivial tree then T has at least one vertex of degree 1)
By our previous theorem, we know there must exist a vertex of degree 1 in T, we will call it v.
Consider if this vertex was removed: $T - v = S$.
Note S has order n now.
It is not possible that removing a vertex from a tree could cause a cycle.
(Must show S is still connected)
Let u and w be any two vertices of S, so that u and w are also vertices of T and there exists a path from u to w in T.
This path cannot include v, since v has degree 1 and thus cannot be in the middle of any path.
Therefore this path still exists in S so S is still connected with the removal of v.

(Easier way would be to say that since v is a leaf node, removing it would not change the connectivity of T. So $T - v = S$. S is connected.)

(Induction) By our induction hypothesis, since S has order n, S has size $n-1$. T has one more vertex than S and one more edge.

(So our claim is proved.) ☐

Can now do the same thing for forests.

Theorem: If G is a forest with k components (trees) and order n then the size of G is $n - k$.

Proof. Let T_1, T_2, ... T_k be the components of G.
Let the order of T_i be n_i.
By the previous theorem for trees, the tree T_i has exactly $n_i - 1$ edges.
The total number of edges in G is the total number of edges in each component,
ie: $\sum_{i=1,k}(n_i - 1) = \sum_{i=1,k}(n_i - k) = n - k$ ☐

Theorem: Any connected graph of order n has size at least $n - 1$

Proof. (Want to use Contradiction)
Note that the theorem is true for $n = 1$ (trivial) and $n = 2$ (one edge).
Suppose that the theorem is false. In other words, suppose there exists graphs of order n with size m where $m < n - 1$ or $m <= n - 2$

Among all graphs that violate the theorem, pick one with the smallest n and least m and call it G (Prove true for $n = 1, 2$).

(Mini Claim): G must have a vertex of degree 1.
Suppose it didn't. Then by summing degrees of the vertices we have:
$2m = \sum_{v \epsilon V(G)} deg(v) \geq 2n$, by the first theory of graph theory.
In this case: $2m \geq 2(m + 2)$ which isn't possible.
So the claim must be true by contradiction.

Call this vertex of degree 1, v.
Let $G_0 = G - v$
See that since v is a leaf node, we still have a connected graph. But the resulting graph G_0 has 1 fewer vertex and 1 fewer edge.
So, we can prove even even smaller ones, and thus G violates the theorem and is the smallest possible counter-example possible. ☐

What we did today in summary:

1. More on working with trees/forests and related proofs
2. Problem solving skills and proof practice
3. Further help with Contradiction/Induction strategies