## Class Notes

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**Theorem 1.** If  $d_1 \ge d_2 \ge d_3 \ge ... \ge d_n$  is a degree sequence, then it is graphical if and only if the sequence  $d_2 - 1, d_3 - 1, ..., d_l(d_1 + 1) - 1, d_l(d_1 + 2), ..., d_n$  is graphical.

Example 1. 5,4,3,3,3,1,1,1,1 is graphical iff 3,2,2,2,0,1,1,1 is graphical

**Proof** of the previous theorem: Must prove both ways since if and only if statement  $(\Leftarrow)$ : Suppose  $d_2 - 1, d_3 - 1, ..., d_{(d_1+1)} - 1, d_{(d_1+2)}, ..., d_n$  is graphical. Then there exists a graph on n - 1 vertices:  $v_2, v_3, ..., v_n$  where  $deg(v_2) = d_2 - 1$ ,  $deg(v_3) = d_3 - 1$ . Produce a new graph on n vertices by adding the vertex  $v_1$  to the hypothesized graph and connecting it to  $v_2, v_3, ..., v_{(d_1+1)}$  and not connecting it to any other vertices. This graph has degree sequence  $d_1, d_2, ..., d_n$ . Therefore  $(\Leftarrow)$  is proved for the theorem.

Now we must prove  $(\Rightarrow)$ : We will prove this by contradiction.

Suppose there exists some sequence  $d_1, d_2, ..., d_n$  which is graphical, but  $d_2 - 1, d_3 - 1, ..., d_{(d_1+1)} - 1, d_{(d_1+2)}, ..., d_n$  is not graphical. Among all graphs with degree sequence  $d_1, d_2, ..., d_n$  take the one where the sums of the degrees of the vertices connected to  $v_1$  is the highest,  $deg(v_1) = d_1$ 

Note: By our assumption,  $v_1$  can not be connected to all of the largest remaining vertices. So,  $v_1$  must be connected to some vertex  $v_r$  where  $deg(v_r) > d_s$  for some s and v is not connected to the vertex  $v_s$  with degree  $d_s$ . So  $v_s$  is connected to some vertex  $v_t$  that  $v_r$  is not connected to.

We can now construct a new graph G' with the same vertices and edges except for the edges  $v_1v_r$  and  $v_sv_t$ . Instead we include the edges  $v_1v_s$  and  $v_rv_t$ . Every vertex in G' has the same degree as in G. If we look at the sums of the degrees of vertices connected to  $v_1$  in G', it is bigger than in G. This is a contradiction with the assumption that G was chosen to have the sum be as big as possible. So, there is a graph that exists where  $v_1$  is connected to all other vertices with largest degrees. Then we can get a graph with degree sequence  $d_2 - 1, d_3 - 1, ..., d_{(d_1 + 1)} - 1, d_{(d_1 + 2)}, ..., d_n$  by removing  $v_1$  from the graph Therefore  $(\Rightarrow)$  is proved.

**Example**: Is 5,4,3,3,2,2,2,1,1,1 graphical?

To solve this we can use the Theorem that we just proved. We know that 5,4,3,3,2,2,2,1,1,1 is graphical if and only if 3,2,2,1,1,2,1,1,1 is graphical We are not certain if 3,2,2,1,1,2,1,1,1 is graphical so we can rearrange the degree sequence and do the computation again.

 $3,2,2,1,1,2,1,1,1 \rightarrow 3,2,2,2,1,1,1,1,1$ 

3,2,2,2,1,1,1,1,1 is graphical if and only if 1,1,1,1,1,1,1,1,1 is graphical.

1,1,1,1,1,1,1,1 is graphical, therefore 5,4,3,3,2,2,2,1,1,1 is graphical.

**Example**: Is 7,7,4,3,3,3,2,1 graphical?

To solve this we can again use the Theorem that we just proved.

We know that 7,7,4,3,3,3,2,1 is graphical if and only if 6,3,2,2,2,1,0 is graphical.

We are not certain if 6,3,2,2,2,1,0 is graphical so we will compute again.

 $6,3,2,2,2,1,0 \rightarrow 2,1,1,1,0,-1$  which is not possible.

Therefore 7, 7, 4, 3, 3, 3, 2, 1 is not graphical.

**Definition:** For graph G of order n with vertices  $v_1, v_2, ..., v_n$  and edges  $e_1, e_2, ..., e_m$  we say the **AdjacencyMatrix** of G is the nxn matrix  $A = [a_i j]$  where  $a_i j = \{1, \text{ if } v_i \text{ is adjacent to } v_j \text{ and } 0 \text{ otherwise}\}$ 

**Definition:** The **IncidenceMatrix** B is an nxm matrix  $B = [b_i j]$  where  $a_i j = \{1, if v_i \text{ is incident to } v_j \text{ and } 0 \text{ otherwise}\}$ 

**Theorem:** The *ij* entry of  $A^k$  is the number of walks of length k from  $v_i$  to  $v_j$