Class Notes

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Theorem 1. *If $d_1 \geq d_2 \geq d_3 \geq ... \geq d_n$ is a degree sequence, then it is graphical if and only if the sequence $d_2 - 1, d_3 - 1, ..., d_(d_1 + 1) - 1, d_(d_1 + 2), ..., d_n$ is graphical.*

Example 1. *5,4,3,3,3,1,1,1,1 is graphical iff 3,2,2,2,0,1,1,1 is graphical*

Proof of the previous theorem: Must prove both ways since if and only if statement $(\Leftarrow)$: Suppose $d_2-1, d_3-1, ..., d_(d_1+1)-1, d_(d_1+2), ..., d_n$ is graphical. Then there exists a graph on $n-1$ vertices: $v_2, v_3, ..., v_n$ where $deg(v_2) = d_2 - 1$, $deg(v_3) = d_3 - 1$. Produce a new graph on n vertices by adding the vertex v_1 to the hypothesized graph and connecting it to $v_2, v_3, ..., v_(d_1 + 1)$ and not connecting it to any other vertices. This graph has degree sequence $d_1, d_2, ..., d_n$. Therefore $(\Leftarrow)$ is proved for the theorem.

Now we must prove $(\Rightarrow)$: We will prove this by contradiction.

Suppose there exists some sequence $d_1, d_2, ..., d_n$ which is graphical, but $d_2 - 1, d_3 - 1, ..., d_(d_1+1)-1, d_(d_1+2), ..., d_n$ is not graphical. Among all graphs with degree sequence $d_1, d_2, ..., d_n$ take the one where the sums of the degrees of the vertices connected to v_1 is the highest, $deg(v_1) = d_1$

Note: By our assumption, v_1 can not be connected to all of the largest remaining vertices. So, v_1 must be connected to some vertex v_r where $deg(v_r) > d_s$ for some s and v is not connected to the vertex v_s with degree d_s. So v_s is connected to some vertex v_t that v_r is not connected to.

We can now construct a new graph G' with the same vertices and edges except for the edges $v_1 v_r$ and $v_s v_t$. Instead we include the edges $v_1 v_s$ and $v_r v_t$. Every vertex in G' has the same degree as in G. If we look at the sums of the degrees of vertices connected to v_1 in G', it is bigger than in G. This is a contradiction with the assumption that G was chosen to have the sum be as big as possible. So, there is a graph that exists where v_1 is connected to all other vertices with largest degrees. Then we can get a graph with degree sequence $d_2 - 1, d_3 - 1, ..., d_(d_1 + 1) - 1, d_(d_1 + 2), ..., d_n$ by removing v_1 from the graph Therefore $(\Rightarrow)$ is proved. ■

Example: Is 5,4,3,3,2,2,2,1,1,1 graphical?
To solve this we can use the Theorem that we just proved.
We know that 5,4,3,3,2,2,2,1,1,1 is graphical if and only if 3,2,2,1,1,2,1,1,1 is graphical

We are not certain if 3,2,2,1,1,2,1,1,1 is graphical so we can rearrange the degree sequence and do the computation again.
3,2,2,1,1,2,1,1,1 $\rightarrow$ 3,2,2,2,1,1,1,1,1
3,2,2,2,1,1,1,1,1 is graphical if and only if 1,1,1,1,1,1,1,1 is graphical.
1,1,1,1,1,1,1,1 is graphical, therefore 5,4,3,3,2,2,2,1,1,1 is graphical.

Example: Is 7,7,4,3,3,3,2,1 graphical?
To solve this we can again use the Theorem that we just proved.
We know that 7,7,4,3,3,3,2,1 is graphical if and only if 6,3,2,2,2,1,0 is graphical.
We are not certain if 6,3,2,2,2,1,0 is graphical so we will compute again.
6,3,2,2,2,1,0 $\rightarrow$ 2,1,1,1,0,-1 which is not possible.
Therefore 7,7,4,3,3,3,2,1 is not graphical.

Definition: For graph G of order n with vertices $v_1, v_2, ..., v_n$ and edges $e_1, e_2, ..., e_m$ we say the **AdjacencyMatrix** of G is the nxn matrix $A = [a_i j]$ where $a_i j = \{1$, if v_i is adjacent to v_j and 0 otherwise$\}$

Definition: The **IncidenceMatrix** B is an nxm matrix $B = [b_i j]$ where $a_i j = \{1$, if v_i is incident to v_j and 0 otherwise$\}$

Theorem: The ij entry of A^k is the number of walks of length k from v_i to v_j