MATH 451 - Class Notes

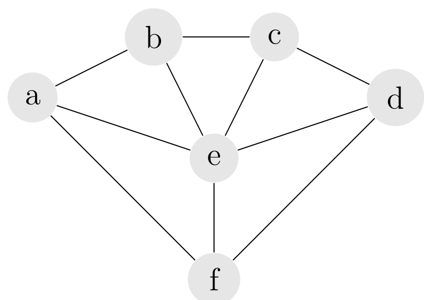
2/1/2018

Scribe: Kathryn Bafford

Defn: A walk is closed if the last vertex in the walk is the same as the first. A walk is open if the last vertex is different from the first.

Defn: A trail which is closed is called a circuit.

Defn: A path which is closed is called a cycle.

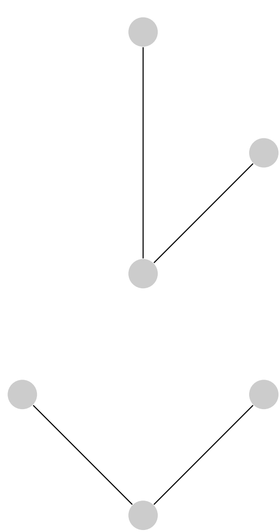


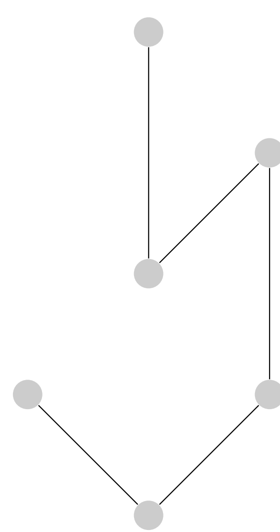
A cycle is (a, b, c, d, e, a)

A k-cycle is a cycle with exactly k-vertices or k-edges. A cycle is odd of its length is odd, and even if its length is even.

Defn: A graph G is connected if for any two vertices u and v in G there exists a walk from u to v.

Defn: A component of a graph is a connected subgraph which is not a subgraph of any larger connected subgraph. Notation: $k(\text{G})$ = number of components of G. $k(G) = 1$ if G is connected and $k(G) > 1$ if G is disconnected.





Same vertices and basic shape, but one is connected and the other is not.

Recall: An equivalence relation is a relation a∼b:

1. If a∼b, then b∼a
2. a∼a for all a
3. if a∼b and b∼c, then a∼c

Proposition: The property of a vertex being connected to another vertex by a walk in a graph is an equivelance relation.

We need to show a walk is reflexive, symmetric, and transitive.

Proof:
Reflexive: The walk from any vertex to itself without going anywhere else connects to any vertex itself.
Symmetric: If a vertex u in out graph is connected to v, then there exists a walk $(u, v_1, v_2, ..., v)$ from u to v. Then $(v, ..., v_2, v_1, u)$ is a walk from v to u.
Transitive: Suppose that a vertex u is connected to vertex v and vertex v is connected to vertex w. Since u is connected to v, there exists a walk $(u, v_1, v_2, ..., v)$ and since v is connected to w, there exists a walk $(v, w_1, w_1, ..., w)$. Therefore the walk $(u, v_1, v_2, ..., v, w_1, w_2, ..., w)$ is a walk from u to w. Therefore, u and w are connected. Q.E.D.

Defn: The distance from vertex u to vertex v in graph G ($d(u, v)$) is the length of the shortest walk from u to v.

If u and v are connected by an edge then the distance is 1.

Defn: A geodesic from u to v is a walk from u to v of length $d(u, v)$.

Proposition: Any geodesic is a path.

Proof: Suppose a geodesic repeats the vertex w on the way from u to v. Then the geodesic looks like:
$(u, v_1, v_2, ..., v_i, w, v_{i+1}, ..., v_j, w, v_{j+1}, ..., v_k, v)$
Then $(u, v_1, v_2, ..., v_i, w, v_{j+1}, ..., v_k, v)$ is a shorter walk. So the original cannot be a geodesic. Q.E.D.

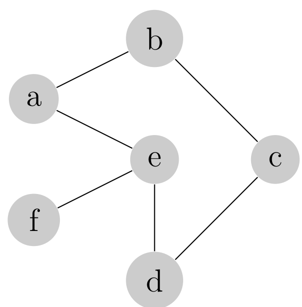
Theorem: If G is a graph of order at least 3 which contains two vertices u and v such that G-u and G-v are both connected, then G is connected.

Proof: Suppose we have a graph G of order at least 3 with two vertices u and v such that G-u and G-v are connected. Let x, y be any two vertices in G. Need to show there is a walk from x to y in G.
Case 1: Suppose x and y are not both u and v. Lets suppose u is neither x nor y. Both x and y are in G-u which is connected so there exists a walk from x to y.
Case 2: Suppose x and y are u and v. Since G has order at least 3, there exists at least one more vertex w which is not u or v. Since G-u is connected, there exists a path from v to w in G-u. Since

G-v is connected, there exists a path from w to u in G-v. Combining these two walks gives a walk from u to v in G. Q.E.D.

Note: The hypothesis that G has order at least 3 is necessary.

Defn: The degree of a vertex is the number of edges in the graph which have the vertex v at one end.
Notation: $degv$



$dega = 2$
$degb = 2$
$degc = 3$
f is a leaf since $degf = 1$

Defn: A vertex with degree 1 is a leaf.

Theorem: For any graph G,

$$\sum_{v \in V(G)} degv = 2m$$

where m is the size of G.

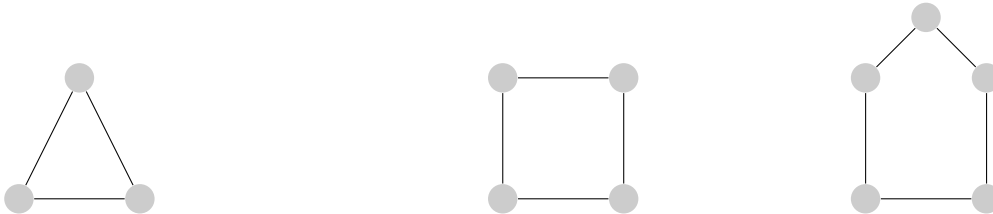
Proof: Count the number of edges, each edge contributes connections of exactly 2 vertices. So summing the degrees of each vertex counts every edge exactly twice. Q.E.D.

Common Graphs:

- If the vertices of a graph of order n can be relabelled as $v_1, v_2, v_3..., v_k$ so the edges of G are exactly $v_1v_2, v_2v_3, ..., v_{k-1}v_k$ then G is called the path. Denoted by P_k.

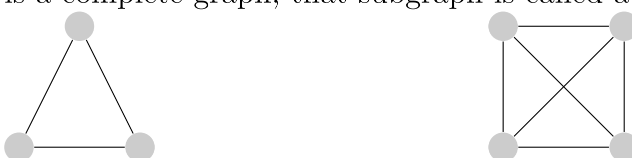
  This is a picture of P_6

- If G is a graph of order k where the vertices can be labelled $v_1, v_2, ..., v_k$ such that the edges are $v_1v_2, v_2v_3, ..., v_{k-1}v_k, v_kv_1$ then G is called the cycle of length k denoted by C_k



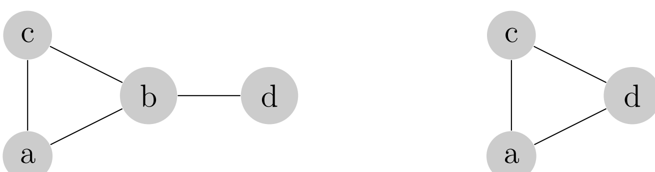
The above graphs are C_3, C_4, and C_5.

- If G is a graph of order n where every vertex is connected to every other vertex by an edge, then G is the complete graph of order n denoted by kn. If a graph G has a subgraph which is a complete graph, that subgraph is called a clique.



The graphs of k_3 and k_4 are shown above.

<u>Defn:</u> The complement of a graph G is denoted by $\overline{G}$ and is the graph with the same vertices as G, but every edge of $\overline{G}$ is not in G and vice-versa.



G and $\overline{G}$ is shown above.