Math 378 - Fall 2024
Homework 3
Due October 15, 2024

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

— G. H. Hardy

Turn in:

(1) Let $\sigma = 6732415$.
  (a) For each of the 6 patterns of length 3, either find an occurence of the pattern in σ, or justify that σ avoids the pattern.
  (b) Find all occurrences of 3412 in σ.

(2) In class we showed that if $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ is a pattern (in one-line notation) and we define $\sigma' = \sigma_k\ldots\sigma_2\sigma_1$ to be the reverse pattern then $\mathrm{Av}_n(\sigma) = \mathrm{Av}_n(\sigma')$ for all n. Now define $\overline{\sigma} = (k+1-\sigma_1)(k+1-\sigma_2)\ldots(k+1-\sigma_k)$, the pattern obtained by "inverting the size" of each element of the pattern. (For example, if $\sigma = 52143$, then $\overline{\sigma} = 14523$.)
  (a) Prove that $\mathrm{Av}_n(\sigma) = \mathrm{Av}_n(\overline{\sigma})$ for all n.
  (b) Combining the result of (a), with the result from class to show that

$$\mathrm{Av}_n(132) = \mathrm{Av}_n(231) = \mathrm{Av}_n(213) = \mathrm{Av}_n(312).$$

  (Note: there is no way to use these facts to prove that $\mathrm{Av}_n(123)$ is equal to the four classes above, that fact is somehow "deeper.")
  (c) Partition the 24 patterns of length 4 into sets whose counts can be proven to be equivalent by the methods of (b).

(3) In the Homework 3 folder in your individual project write:
  (a) A function which uses recursion, called `BinomRec(n,k)` which computes the binomial coefficient $\binom{n}{k}$, using only the following facts about binomial coefficients (In particular it shouldn't use any factorials or multiplications):
    (i) $\binom{0}{0} = 1$
    (ii) $\binom{n}{k} = 0$ if $k < 0$ or $k > n$.
    (iii) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ (Pascal's identity)
  (b) A generator called `BinomialRow(n)` that yields the binomial coefficients in the n-th row of Pascal's triangle one at a time. (Use the function from part a.)
  (c) Measure how long (in seconds) it takes your code from part a to compute the "middle" binomial coefficients $\binom{2n}{n}$ for $n = 1, 2, 3, 4\ldots$ as far as you can go. Based on these times, make a prediction for how long it would take your code to compute $\binom{100}{50}$ and $\binom{2000}{1000}$. (Hint: you might want to checkout SageMath's `timeit` command.)