Math 378 - Fall 2024
Homework 1
Due September 10, 2024

In mathematics you don't understand things. You just get used to them.

— John von Neumann

Turn in:

(1) Let $\sigma = 7135426$ and $\tau = 4513762$ (In one-line-notation).
  (a) Write both σ and τ in cycle notation.
  (b) Compute σ^2 and τ^2.
  (c) Compute $\sigma \circ \tau$ and $\tau \circ \sigma$. (Remember we take the left-to-right convention, not right-to-left.)
  (d) The **order** of a permutation σ is the least (positive) number n such that σ^n is the identity permutation. Find the order of both σ and τ.

Note: a **transposition** is a permutation that swaps two elements (a cycle of length 2).

(2) Fix an integer n. Our goal is to prove that the following proceedure (the Fisher–Yates algorithm) produces each permutation σ of n with equal likelihood.

> **The Fisher–Yates algorithm:** For each integer $i = 1, 2, \ldots n$, pick a random integer $a_i \in \{i, i+1, \ldots, n\}$. Then define
>
> $$\sigma = (1, a_1) \circ (2, a_2) \circ (3, a_3) \cdots (n, a_n).$$

  (a) Count the number of possible choices for the sequences $a_1, a_2, \ldots, a_n$ that satisfy the condition $a_i \in \{i, i+1, \ldots, n\}$.
  (b) Fix $n = 5$. Find values of a_1, a_2 etc satisfying this condition so that $\sigma = 43152$.
  (c) Show that every permutation π of n can be produced by this process. To do this, fix a permutation π of n written (in one line notation) as $\pi = \pi_1\pi_2 \cdots \pi_n$ and describe how to pick the values of a_1, a_2 etc so that the permutation σ described above is equal to π.
  (d) Now prove that by combining the observations of (a) and (c) we can conclude that the Fisher-Yates algorithm is equally likely to produce any permutation.