Series Solutions and Ordinary Points

Definitions and Theorems

Definition 1. A function $f(x)$ is analytic at a point x_0 if it can be represented as a power series centered at x_0 with a radius of convergence greater than 0,

Definition 2. A point x_0 is called an ordinary point of the differential equation $y'' + P(x)y' + Q(x)y = 0$ if both $P(x)$ and $Q(x)$ are analytic at x_0. A point that is not ordinary is singular.

Theorem 1. If x_0 is an ordinary point of the differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ then we can always find two linearly independent power series solutions centered at x_0, i.e $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$, which converges on some interval $|x - x_0| < R$ where R is the distance from x_0 to the nearest singular point.

Remark 1. Note that we must take into consideration complex numbers as singular points.

Check for Understanding

Example 1. Determine if the following differential equations contain any singular points:

- $y'' + y' = 0$
- $2y'' + 3y' - 5y = 0$
- $y'' + xy' + \ln(x)y = 0$
- $xy'' + y' + xy = 0$

Example 2.

Consider a differential equation of the form in Theorem 1 but for, $0 \leq i \leq 2$, $a_i(x)$ is a polynomial. When would said differential equation contain any singular points? (Check 14:00 of recorded lecture 04/27/2020)

Example 3. Find the min radius of convergence guaranteed by Theorem 1 for $x_0 = 0$ for the differential equation $(x^2 - 2x - 5)y'' - (xy' + y) = 0$. (Solution is at 23:17 of 04/27/2020 recorded lecture).

Example 4. Consider Airy's Equation, $y'' + xy = 0$. Apply Theorem 1 and find series solutions for $x_0 = 0$. (Solution is at 23:00 of 04/29/2020 recorded lecture)

Example 5. Find the first 6 terms of a series solution to $(x - 1)y'' - xy' + y = 0$ satisfying $y(0) = -2$ and $y'(0) = 6$ centered at $x_0 = 0$ (Hint: First find radius of convergence by determining if differential equation has any singular points). (Solution is at 1:02:00 of 04/29/2020 recorded lecture).

Example 6. Consider the differential equation $y'' + \sin(x)y = 0$. Find the first three terms of a solution centered at $x_0 = 0$ and initial value $y(0) = 1$ and $y'(0) = 2$ (Hint: Can we represent $\sin(x)$ as a convergent power series at $x_0 = 0$?). (Explanation of Solution begins at 1:04:00 of 04/29/2020 recorded lecture).