## Series Solutions and Ordinary Points

## Definitions and Theorems

Definition 1. A function $f(x)$ is analytic at a point $x_{0}$ if it can be represented as a power series centered at $x_{0}$ with a radius of convergence greater than 0 ,
Definition 2. A point $x_{0}$ is called an ordinary point of the differential equation $y^{\prime \prime}+P(x) y^{\prime}+Q(x) y=0$ if both $P(x)$ and $Q(x)$ are analytic at $x_{0}$. A point that is not ordinary is singular.
Theorem 1. If $x_{0}$ is an ordinary point of the differential equation
$a_{2}(x) y^{\prime \prime}+a_{1}(x) y^{\prime}+a_{0}(x) y=0$ then we can always find two linearly independent power series solutions centered at $x_{0}$, i.e $y=\sum_{n=0}^{\infty} c_{n}\left(x-x_{0}\right)^{n}$, which converges on some interval $\left|x-x_{0}\right|<R$ where $R$ is the distance from $x_{0}$ to the nearest singular point.
Remark 1. Note that we must take into consideration complex numbers as singular points.

## Check for Understanding

Example 1. Determine if the following differential equations contain any singular points:

- $y^{\prime \prime}+y^{\prime}=0$
- $2 y^{\prime \prime}+3 y^{\prime}-5 y=0$
- $y^{\prime \prime}+x y^{\prime}+\ln (x) y=0$
- $x y^{\prime \prime}+y^{\prime}+x y=0$


## Example 2.

Consider a differential equation of the form in Theorem 1 but for, $0 \leq i \leq 2, a_{i}(x)$ is a polynomial. When would said differential equation contain any singular points? (Check 14:00 of recorded lecture $04 / 27 / 2020$ )
Example 3. Find the min radius of convergence guaranteed by Theorem 1 for $x_{0}=0$ for the differential equation $\left(x^{2}-2 x-5\right) y^{\prime \prime}-\left(x y^{\prime}+y\right)=0$. (Solution is at 23:17 of 04/27/2020 recorded lecture).
Example 4. Consider Airy's Equation, $y^{\prime \prime}+x y=0$. Apply Theorem 1 and find series solutions for $x_{0}=0$. (Solution is at 23:00 of 04/29/2020 recorded lecture)
Example 5. Find the first 6 terms of a series solution to $(x-1) y^{\prime \prime}-x y^{\prime}+y=0$ satisfying $y(0)=-2$ and $y^{\prime}(0)=6$ centered at $x_{0}=0$ (Hint: First find radius of convergence by determining if differential equation has any singular points). (Solution is at 1:02:00 of 04/29/2020 recorded lecture).
Example 6. Consider the differential equation $y^{\prime \prime}+\sin (x) y=0$. Find the first three terms of a solution centered at $x_{0}=0$ and initial value $y(0)=1$ and $y^{\prime}(0)=2$ (Hint: Can we represent $\sin (x)$ as a convergent power series at $x_{0}=0$ ?). (Explanation of Solution begins at 1:04:00 of $04 / 29 / 2020$ recorded lecture).

