Differential Equations April 20th Notes

Avery Schweitzer

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1 Power Series Solutions

Power series is a series of the form of $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a.
$\sum_{n=0}^{\infty}$ is shorthand for $\lim_{l\to\infty}\sum_{n=0}^{l}$. The limit may not exist, i.e. the series may not be convergent, for all values of x.

The interval of convergence, called I, is the interval on which the series is convergent. $I = (a-R, a+R)$. R is the radius of convergence.

At endpoints of the intervals the series may converge or diverge.

Absolute Convergence: If $\sum_{n=0}^{\infty} |_n(x-a)^n|$ converges then the series is absolutely convergent. Absolute convergence implies convergence.

Every term of the power series is differentiable within the interval of convergence. $\frac{d[c_n(x-a)^n]}{dx} = nc_n(x-a)^{n-1}$ so the derivative of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = \sum_{n=0}^{\infty}(n-1)c_{n-1}(x-a)^n$.

Ratio test: Main tool to find the interval of convergence.

$L = \lim_{n\to\infty} \mid \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \mid = \mid x-a \mid \lim_{n\to\infty} \mid \frac{c_{n+1}}{c_n} \mid$
If $L < 1$ then the series converges absolutely at x.
If $L > 1$ then the series diverges at x.
If $L = 1$ then inconclusive

To find R, find the values of x such that $L < 1$.

Ex 1.1. $\sum_{n=0}^{\infty} \frac{2^n}{n}(x-2)^n$ centered at 2
$L = \mid x-2 \mid \lim_{n\to\infty} \mid \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} \mid = \mid x-2 \mid \lim_{n\to\infty} \mid \frac{2n}{n+1} \mid = 2 \mid x-2 \mid$

$2 \mid x-2 \mid < 1$ holds for $x \in (3/2, 5/2)$ so $R = 1/2$

check endpoints of interval
$x = 3/2$ implies $\sum \frac{2^n}{n}(3/2-2)^n = \sum \frac{(-1)^n}{n}$ which converges.
$x = 5/2$ implies $\sum \frac{2^n}{n}(5/2-2)^n = \sum \frac{(2)^n}{n}(1/2^n) = \sum 1/n$ which diverges.

So $I = [3/2, 5/2)$.

Identity Property
If $\sum_{n=0}^{\infty} c_n(x-a)^n = 0$ on I, then $c_n = 0$.

Taylor Series
$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Maclaurin Series is centered at 0.
$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$
Infinitely differentiable functions can be represented by the Maclaurin series.

Addition of power series: make sure center and exponents match.

Ex 1.2. $S = \sum_{n=2}^{\infty} n(n-1)c_n(x)^{n-2} + \sum_{n=0}^{\infty} c_n(x)^{n+1}$

$= 2c_2 + \sum_{n=3}^{\infty} n(n-1)c_n(x)^{n-2} + \sum_{n=0}^{\infty} c_n(x)^{n+1}$

$= 2c_2 + \sum_{n=0}^{\infty}(n+3)(n+2)c_{n+3}(x)^{n+1} + \sum_{n=0}^{\infty} c_n(x)^{n+1}$

$= 2c_2 + \sum_{n=0}^{\infty}((n+3)(n+2)c_{n+3} + c_n)(x)^{n+1}$

Ex 1.3. Use power series to find a solution to $y' + y = 0$.

$y = \sum_{n=0}^{\infty} c_n x^n$

$y' = \sum_{n=1}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$

$y' + y = \sum_{n=0}^{\infty}((n+1)c_{n+1} + c_n)x^n = 0$

Use Identity Property.
$(n+1)c_{n+1} + c_n = 0$ so $c_{n+1} = -\frac{c_n}{n+1}$

$c_1 = -c_0$
$c_2 = -c_1/2 = c_0/2$
$c_3 = -c_2/3 = -c_0/6$
$c_4 = -c_3/4 = c_0/24$

$c_n = (-1)^n \frac{c_0}{n!}$ so $y = \sum_{n=0}^{\infty} c_0 \frac{(-1)^n}{n!} x^n = c_0 e^{-x}$