Differential Equations April 13th Notes

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1 Systems of Equations

Ex 1.1. 2 tanks: Each holds 24 liters.
Tank 1

- Starts with 2 lbs of salt
- inflow of 6 liters per minute of pure water
- inflow of 2 liters per minute from tank 2
- outflow of 8 liters per minute

Tank 2

- Starts with no salt
- inflow of 8 liters per minute from tank 1
- outflow of 8 liters per minute

$\frac{dx}{dt} = 6$ liters per minute $*$ 0 lbs of salt per liter + 2 liters per minute $* \, y(t)/24$ lbs of salt per liter $-$ 8 liters per minute $* \, x(t)/24$ lbs of salt per liter $= y(t)/12 - x(t)/3$.
$\frac{dx}{dt} = y(t)/12 - x(t)/3$.
Likewise, $\frac{dy}{dt} = x(t)/3 - y(t)/3$.

We want to eliminate one variable.
We have $(D + 1/3)x - y/12 = 0$ and $(D + 1/3)y - x/3 = 0$.
$1/3(D + 1/3)x - y/36 = 0$
$(D + 1/3)^2 y - 1/3(D + 1/3)x = 0$
$(D + 1/3)^2 y - y/36 = 0$
$D^2 y + \frac{2}{3} Dy + y/12 = 0$
$(D^2 + \frac{2}{3} D + \frac{1}{2})y = 0)$

Find Auxiliary Equation
$m^2 + \frac{2}{3}m + \frac{1}{12} = 0$
$(m + 1/2)(m + 1/6) = 0$

$m = -1/2, -1/6$
$y(t) = c_1e^{-t/2} + c_2e^{-t/6}$

Now we want to find $x(t)$.
$\frac{dy}{dt} = x(t)/3 - y(t)/3.$
$x(t) = 3\frac{dy}{dt} + y$

We plug in y to get $x = \frac{-1}{2}c_1e^{-t/2} + \frac{1}{2}c_2e^{-t/6}$

Solve for c_1, and c_2 using $x(0) = 2$ and $y(0) = 0$ to get $c_1 = -2$ and $c_2 = 2$. So

$x(t) = e^{-t/2} + e^{-t/6}$ and $y(t) = -2e^{-t/2} + 2e^{-t/6}$

Ex 1.2. $x' - 3x + 4y = 1$ so $(D - 3)x + 4y = 1$.
$-4x + y' + 7y = 10t$ so $(D + 7)y - 4x = 10t$.

Eliminate One Variable
$4(D - 3)x + 16y = 4$
$(D - 3)(D + 7)y - (D - 3)4x = (D - 3)10t = 10 - 30t.$

Add together to get:
$(D - 3)(D + 7)y + 16y = 14 - 30t$
$y'' + 4y' - 5y = -14 - 30t$

Find Complimentary Solution
Aux: $m^2 + 4m - 5 = 0$
$(m - 1)(m + 5) = 0$
$y_c = c_1e^t = c_2e^{-5t}$

Find Particular Solution
$y_p = At + B$
$y_p' = A$ and $y_p'' = 0$ so $y_p'' + 4y_p' - 5y_p = -5At + 4A - 5B$.
So $-5At + 4A - 5B = 14 - 30t$. Then $A = 6$ and $B = 2$.
$y = c_1e^t + c_2e^{-5t} + 6t + 2$

Now we can find $x(t)$ by plugging y and y' into $(D + 7)y - 4x = 10t$ to get $x = 2c_1e^t + \frac{1}{2}c_2e^{-5t} + 8t + 5$.

2 Nonlinear Equations of Higher Order

Sum of homogeneous solutions is not necessarily a homogeneous solution.

Most of these can't be solved analytically. Can solve if either independent variable x or y-term is missing. $F(x, y', y'') = 0$ or $F(y, y', y'') = 0$. Use substitution $u = y'$ to make a first order equation. $y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u'y' = u'u$.

Ex 2.1. $y + y'' = y(y')^2$
$y + u'u = yu^2$
$u' = \frac{yu^2 - y}{u} = y(u - 1/u)$
$y = \frac{u'}{u - 1/u}$

Take the integral of both sides to get $y^2 = ln(u^2 - 1) + c$
$ke^{y^2} = u^2 - 1$ where $k = (e^c)^{-1}$
$u = \sqrt{ke^{y^2} + 1}$
$y' = \sqrt{ke^{y^2} + 1}$

We dont think this is solvable.