MATH 314 Spring 2020 - Class Notes

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Summary: In this class, we learned how to solve non-homogeneous linear DE that weren't solvable using undetermined coefficients method by introducing a new method, Variation of Parameters.

First get the DE into Standard Form: Y"+Py'+Qy=F(x) then,

Method: "Guess"

$Y_p = u_1(x)y_1 + u_2(x)y_2$

where $u_1(x)$ $u_2(x)$ are some functions

Goal: To find $u_1(x)$ $u_2(x)$ that satisfies non-homogeneous

$Y_p = u_1y_1 + u_2y_2$
$Y_p' = u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2$
$Y_p' = u_1y_1' + u_2y_2' + \cancel{[u_1'y_1 + u_2'y_2]}$

Assume $[u_1'y_1 + u_2'y_2] = 0$
$Y_p' = u_1y_1' + u_2y_2'$

$Y_p'' = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'$
Plug the values back in our standard form DE ,

Y"+Py'+Qy=F(x)

$(u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2') + P(u_1y_1' + u_2y_2') + Q(u_1y_1 + u_2y_2)$

$u_1\cancel{(y_1'' + Py_1' + Qy_1)} + u_2\cancel{(y_2'' + Py_2' + Qy_2)} + u_1'y' + u_2'y_2' = F(x)$ since y1 and y2 are solutions to the homogeneous DE

$(y_1'' + Py_1' + Qy_1) = 0$ and $(y_2'' + Py_2' + Qy_2) = 0$

thus,
$u_1'y' + u_2'y_2' = F(x)$

Conditions:
$u_1'y_1 + u_2'y_2 = 0$ ....Assumption–(1)
$u_1'y_1' + u_2'y_2' =$ F(x).................... What we derived–(2)

<u>**Solve for u_1' and u_2':**</u>

Multiply (1) by y_2'..............$u_1'y_1y_2 + \cancel{u_2'y_2y_2'} = 0$

Multiply (2) by y_2.......... $u_1'y_1'y_2$+$\cancel{u_2'y_2'y_2}$=F(x)y_2

After subtracting (1) from (2) we end up with,

$u_1'y_1y_2' - u_1y_1'y_2 = -F(x)y_2$

$u_1' = \frac{-F(x)y_2}{y_1y_2' - y_1'y_2}$

$u_1' = \frac{-F(x)y_2}{W}$

$$\text{Where, W = Wronskian}(y_1, Y_2) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$
$$= y_1y_2' - y_1'y_2$$

In the same way we can solve for u_2' $u_2' = \frac{f(x)y_1}{w}$

$u_1 = -\int \frac{f(x)y_2}{w}$

$u_2 = \int \frac{f(x)y_1}{w}$

Example:

Find a particular solution to:

$2y'' - 4y' + 2y = 2\frac{e^x}{x}$

First change the DE into standard form,
$y'' - 2y' + y = \frac{e^x}{x}$

Find y_1 and y_2 solution,
Auxiliary equation:

$m^2 - 2m + 1 = 0$
(m-1)(m-1)=0
therefore, $y_1 = e^x$ and $y_2 = e^{2x}$

$\mathrm{W}(y_1, Y_2) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^x & xe^x \\ e^{x\prime} & e^x + xe^x \end{bmatrix} = e^{2x}$
$\mathrm{u}_1 = -\int \frac{\frac{e^x}{x} y_2}{e^{2x}} dx = -\int \frac{\frac{e^x}{x}(xe^x)}{e^{2x}} dx = -\int 1 dx = -x$

$u_2 = \int \frac{\frac{e^x}{x} y_1}{e^{2x}} dx = \int \frac{\frac{e^x}{x}(e^x)}{e^{2x}} dx = \int \frac{1}{x} dx = ln|x|$

Particular solution:
$\mathrm{Y}_p = -xe^x + ln|x|xe^x$

General solution:
$Y = Y_p + c_1 e^x + c_2 xe^x$