MATH 374 Fall 2020 - Class Notes

3/09/2020

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Summary: Today's Class covered the superposition principle for both homogeneous and non-homogeneous equations and the Wronskian in order to determine linear independence.

Notes:
General Form for an n-th order linear differential equation:

$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y = g(x)$

Call this a homogeneous equation if g(x) = 0

Ex:

1. $y'' + sin(x)y' + cos(x)y = 0$ (homogeneous)
2. $y'' + cos(x)y = sin(x)$ (non-homogeneous)

D - derivative operator

$Dy = y' = \frac{dy}{dx}$

$(D^2 + D + 1)y = D^2y + Dy + y = y'' + y' + y$

Sometimes we write $L(y) = x^2 + 3$ where $L = D^2 + D$ means

$(D^2 + D)(y) = x^2 + 3$

or

$y'' + y' = x^2 + 3$

Theorem (Superposition Principle)
If $y_1, y_2, ..., y_k$ are solutions to an n-th order linear homogeneous equation then so is any linear combination of them.

$C_1y_1 + C_2y_2 + ... + C_ky_k$ is also a solution to this differential equation

Ex: $y'' - 9y = 0$
Check that $y_1 = e^{3x}$ is a solution, as is $y_2 = e^{-3x}$

$y_1 = e^{3x}$
$y_1 = 3e^{3x}$
$y_1 = 9e^{3x}$

$y_1'' - 9y = 9e^{3x} - 9e^{3x} = 0\checkmark$

$y_2 = e^{-3x}$
$y_2 = -3e^{-3x}$
$y_2 = 9e^{-3x}$

$y_2'' - 9y = 9e^{-3x} - 9e^{-3x} = 0\checkmark$

Our Superposition principle says that $C_1e^{3x} + C_2e^{-3x}$ is also a solution to this differential equation for any constants C_1 and C_2

A family of equations $f_1(x), f_2(x), ..., f_k(x)$ is linearly dependent on an interval I if there exist constants $C_1, C_2, ..., C_k$ not all zero such that $C_1f_1(x) + C_2f_2(x) + ... + C_kf_k(x)$ is zero for all x in I.

Ex:

$f_1(x) = sin^2(x)$
$f_2(x) = cos^2(x)$
$f_3(x) = tan^2(x)$
$f_4(x) = sec^2(x)$

Are these linearly dependent on $(\frac{-\pi}{2}, \frac{\pi}{2})$

$sin^2(x) + cos^2(x) = 1$
$tan^2(x) - sec^2(x) = -1$

take $C_1 = C_2 = C_3 = 1$ and $C_4 = -1$

$sin^2(x) + cos^2(x) + tan^2(x) - sec^2(x) = (1) + (-1) = 0\checkmark$

Any family of equations that is not linearly dependent is called linearly independent.

Theorem:

If we have an n-th order linear homogeneous differential equation the nit will have n linearly independent solutions $y_1, y_2, ..., y_n$ and <u>any</u> solution to this differential equation will be a linear combination of these solutions.

Suppose:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y = g(x)$$

is an nth order linear differential non-homogeneous equation with a particular solution y_p. If $y_1, y_2, ..., y_n$ are solutions to the homogeneous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y = 0$$

then

$C_1y_1 + C_2y_2 + ... + C_ny_n + y_p$ is a solution to the original non-homogeneous equation.

Complementary Solution: $C_1y_1 + C_2y_2 + ... + C_ny_n$
Particular Solution: y_p

Def: The <u>Wronskian</u> of a family of equations $y_1, y_2, ..., y_n$ is the determinant of the matrix

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \end{vmatrix}$$

The family $y_1, y_2, ..., y_n$ is linearly dependent on I if and only if the Wronskian is 0 on I.

Ex: $y'' - 9y = 0$

$y_1 = e^{3x}$ and $y_2 = e^{-3x}$ were both solutions. Are these solutions linearly independent?

Reminder Determinant of a 2x2 Matrix $\begin{vmatrix} A & B \\ C & D \end{vmatrix}$ =AD-BC

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} \mathrm{e}^{3x} & \mathrm{e}^{-3x} \\ 3\mathrm{e}^{3x} & -3\mathrm{e}^{-3x} \end{vmatrix}$$

$$\begin{aligned} W(e^{3x}, e^{-3x}) &= e^{3x}(-3e^{-3x}) - e^{-3x}(3e^{3x}) \\ &= -3 - 3 = -6 \neq 0 \end{aligned}$$

These equations are linearly independent

What about $y'' + 9y = 0$?

$y_1 = sin(3x)$ and $y_2 = cos(3x)$
$y_1' = 3cos(3x)$
$y_1'' = -9sin(3x)$

$y'' + 9y = -9sin(3x) + 9sin(3x) = 0\checkmark$

Are $sin(3x) and cos(3x)$ linearly independent?

$$\begin{aligned} W(sin(3x), cos(3x)) &= \begin{vmatrix} sin(3x) & cos(3x) \\ 3cos(3x) & -3sin(3x) \end{vmatrix} \\ &= -3sin^2(3x) - 3cos^2(3x) \\ &= -3(sin^2(3x) - cos^2(3x)) \\ &= -3 \end{aligned}$$

So $sin(3x)$ and $cos(3x)$ are linearly independent. All solutions to $y'' + 9y = 0$ are of the from $y = C_1 sin(3x) + C_2 cos(3x)$.

Theorem (Superposition principle for non-homogeneous differential equations)

If y_{p_1} is a particular solution to

$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y = g_i(x)$

for some family $g_1(x)$,

then $y_{p_1} + y_{p_2} + ... + y_{p_k}$ is a particular solution to

$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y = g_1(x) + g_2(x) + ... + g_1(k)$

Suppose you want to solve $y'' + 3y' - 7y = 1 + x$. Find a solution y_1 to $y'' + 3y' - 7y = 1$ and a solution $y_2$to $y'' + 3y' - 7y = x$.

Then $y_1 + y_2$ is a solution to the original differential equation.