Exact DEs

An equation $M(x,y)dx + N(x,y)dy = 0$ is exact if <M(x,y), N(x,y)> is conservative.

If there exists a function F(x,y)=Z such that $M(x,y) = \frac{df}{dx}$ and $N(x,y) = \frac{df}{dy}$
Test: this form is exact if and only if: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

A differential equation is exact if and only if its solutions are the levelcurves of the surface $Z = F(x,y)$

Note: $M(x,y)dx + N(x,y)dy = 0$
Solve for $\frac{dy}{dx}$
$N(x,y)dy = -M(x,y)dx$
$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$
We can solve by integrating both sides
$y' = \frac{3x^2y^2-y^3}{3xy^2-2x^3y} = \frac{dy}{dx}$
$(3x^2y^2 - y^3)dx = (3xy^2 - 2x^3y)dy$
$(3x^2y^2 - y^3)dx - (3xy^2 - 2x^3y)dy = 0$

Test for exactness:
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3x^2y^2 - y^3)$
$\frac{\partial M}{\partial y} = 6x^2y - 3y^2$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-3xy^2 + 2x^3y)$
$\frac{\partial N}{\partial x} = -3y^2 + 6x^2y$

$-3y^2 + 6x^2y = 6x^2y - 3y^2$
Use the method of exact equations to solve
Integrate:

$\int M(x,y)dx = \int(3x^2y^2 - y^3)dx$
$\int M(x,y)dx = x^3y^2 - xy^3 + C$

$\frac{\partial}{\partial y}(x^3y^2 - xy^3 + C(y)) = 2x^3y - xy^2 + C'(y)$
Set equal to N(x,y)
$2x^3y - xy^2 + C'(y) = 2x^3y - xy^2$
$C'(y) = 0$
$C'(y) = c$

$F(x,y) = x^3y^2 - xy^3 + C$
$C = x^3y^2 - xy^3$

Substitution

Various tricks for things we can substitute to get an equation into a form we can solve.
Homogeneous equations:

A function F(x,y) is called homogeneous if $F(tx, ty) = t^{\alpha} F(x, y)$

A DE is homogeneous if $y' = F(x, y)$ where F(x,y) is homogeneous or for an exact equation if both M(x,y) and N(x,y) are homogeneous

Ex.
$F(x, y) = x^2 y^3 + y^4 x$
$F(tx, ty) = (tx)^2 (ty)^3 + (ty)^4 (tx)$
$F(tx, ty) = t^2 x^2 t^3 y^3 + t^4 y^4 tx$
$F(tx, ty) = t^5 x^2 y^3 + t^5 y^4 x$
$F(tx, ty) = t^5 (x^2 y^3 + y^4 x)$
$F(tx, ty) = t^5 (F(x, y))$

Ex.
$F(x, y) = x + e^y$ this is not homogeneous because,
$F(tx, ty) = tx + e^{ty}$ which does not allow us to factor out a t

If we have a homogeneous DE we make the substitution:
$u = \frac{y}{x}, y = ux$
$y' = \frac{d}{dx}(ux)$
$y' = u'(x) + u(1)$
$y' = u'(x) + u$