# Math 374 Notes

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## 1 Review of Linear Differential Equations

### 1.1 Formulas to Know

Standard Form:

$$y' + P(x)y = f(x)$$

Integrating Factor:

$$\mu(x) = e^{\int P(x)\,dx}$$

### 1.2 Find the solution to $x\frac{dy}{dx} - 4y = x^6e^x$

Step 1: Convert to standard form by dividing across $x$

$$\frac{dy}{dx} - \left(\frac{4}{x}\right)y = x^5e^x$$

Step 2: Find integrating factor

$$\mu(x) = e^{\int -\frac{4}{x}\,dx}$$

$$\mu(x) = e^{-4\ln|x|} = \left(e^{\ln|x|}\right)^{-4} = |x|^{-4}$$

$$\mu(x) = x^{-4}$$

Step 3: Multiply DE by integrating factor $\mu(x) = x^{-4}$

$$\frac{dy}{dx}\left(\frac{1}{x^4}\right) - \left(\frac{4}{x^5}\right)y = xe^x$$

Step 4: "Undo" product rule and integrate both sides of the equation with respect to $x$

$$\left[\left(\frac{1}{x^4}\right)y\right]' = xe^x$$

$$\int\left[\left(\frac{1}{x^4}\right)y\right]'\,dx = \int xe^x\,dx$$

Use integration by parts to integrate $\int xe^x\,dx$

$$u = x,\ du = dx,\ dv = e^x\,dx,\ v = e^x$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

The integrated equation becomes

$$x^{-4}y = xe^x - e^x + C$$

$$y = x^5e^x - x^4e^e + Cx^4$$

### 1.3 Find the solution to $\frac{dy}{dx} - y\tan(x) = \sin(x)$ given the initial condition $y(0) = 1$

Step 1: Find integrating factor

$$\mu(x) = e^{-\int \tan(x)\,dx}$$

$$\mu(x) = e^{-(-\ln|\cos(x)|)} = |\cos(x)|$$

$$\mu(x) = \cos(x)$$

Step 2: Multiply DE by integrating factor $\mu(x) = \cos(x)$ and simplify

$$\cos(x)\frac{dy}{dx} - y\tan(x)\cos(x) = \sin(x)\cos(x)$$

$$\cos(x)\frac{dy}{dx} - y\sin(x) = \sin(x)\cos(x)$$

Step 3: "Undo" product rule and integrate both sides of the equation with respect to $x$

$$[\cos(x)y]' = \sin(x)\cos(x)$$

$$\int [\cos(x)y]'\,dx = \int \sin(x)\cos(x)\,dx$$

Use u-substitution to integrate $\int \sin(x)\cos(x)\,dx$

$$u = \sin(x),\ du = \cos(x)\,dx$$

$$\int \sin(x)\cos(x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C$$

The integrated equation becomes

$$\cos(x)y = \frac{1}{2}\sin^2(x) + C$$

$$y = \frac{\sin^2(x)}{2\cos(x)} + \frac{C}{\cos(x)}$$

$$y = \frac{1}{2}\sin(x)\tan(x) + C\sec(x)$$

Step 4: Use initial condition $y(0) = 1$ to solve for C

$$1 = \frac{1}{2}\sin(0)\tan(0) + C\sec(0)$$

$$1 = (0) + (1)C \;\rightarrow\; C = 1$$

$$y = \frac{1}{2}\sin(x)\tan(x) + \sec(x)$$

# 2 Introduction of Exact Differential Equations

## 2.1 Background Information

For a surface $z = F(x, y)$, the level curves of the surface are solutions to $C = F(x, y)$ for some C.
The Total Differential of $F(x, y)$ is

$$dz = \frac{\delta F}{\delta x}\,dx + \frac{\delta F}{\delta y}\,dy$$

A differential $M(x, y)\,dx + N(x, y)\,dy$ is *exact* if there exists a function $F(x, y)$ such that

$$M(x, y) = \frac{\delta F}{\delta x} \quad \text{and} \quad N(x, y) = \frac{\delta F}{\delta y}$$

A vector field $\langle M(x, y)\,,\,N(x, y)\rangle$ is conservative if the differential is exact and if

$$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

If a differential is exact, work backward to solve for $F(x, y)$

## 2.2 Verify that $(2x + y)\,dx + (2y + x)\,dy$ is exact and find $F(x, y)$

Step 1: Find $\frac{\delta M}{\delta y} and \frac{\delta N}{\delta x}$ to check if the differential is exact

$$M(x, y) = 2x + y, \quad \frac{\delta M}{\delta y} = 1$$

$$N(x, y) = 2y + x, \quad \frac{\delta N}{\delta x} = 1,$$

$$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x} = 1,$$

so the differential is exact.
Step 2: Integrate $N(x, y)$ to get an expression for $F(x, y)$ that has the constant in terms of $x$

$$\int N(x, y)\,dy = \int (2y + x)\,dy$$

$$F(x, y) = y^2 + xy + C(x)$$

Step 3: Differentiate $F(x, y)$ with respect to $x$. Set the resulting expression equal to $M(x, y)$ to solve for $C'(x)$

$$M(x, y) = \frac{\delta F}{\delta x} = 0 + y + C'(x)$$

$$2x + y = y + C'(x)$$

$$C'(x) = 2x$$

Step 4: Integrate $C'(x)$ with respect to $x$ to find $C(x)$

$$\int C'(x)\,dx = \int 2x\,dx$$

$$C(x) = x^2 + C$$

Step 5: Substitute $C(x)$ into the expression for $F(x, y)$ that has the constant in terms of $x$

$$F(x, y) = y^2 + xy + C(x)$$

$$F(x, y) = y^2 + xy + x^2 + C$$