MATH 374 Spring 2020 - Class Notes

1/27/2020

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Summary: In this class, we discussed the definition of a differential equation, different notations for derivatives, how to categorize differential equations, and questions of importance regarding solutions to a differential equation.

Differential Equation Definition:

- A differential equation is an equation involving a function y along with its higher derivatives.

Derivative Notations:

- Liebniz notation: $\frac{dy}{dx}$
- Prime notation: y'
- Euler's notation: $D(y)$, emphasizes derivation as an operation.
- $f(x) = y, f_x$. take the derivative f with respect to x. Used more when several independent variables exist.
- Sometimes we will also write differential equations in differential form. $m(x,y)\frac{dy}{dx} = n(x,y) \Rightarrow m(x,y)dy = n(x,y)dx$

Categorizing differential equations:

- Differential equations can be categorized according to three criteria:
  1. Type (Ordinary differential equations (O.D.E.), learned in this course, and partial differential equations (P.D.E).
  2. Order
  3. Linearity

Ordinary Differential Equations

- O.D.E.s only involve derivatives with respect to one independent variable.

Examples:

$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + xy = 0$

$\frac{dy}{dt} + t^2 = y$

Partial Differential Equations

- P.D.E.s have derivatives with respect to multiple independent variables.
- These types are much harder in general.

Examples:

$\frac{du}{dx} - \frac{du}{dy} = 0$

Schrödinger's Equation: $\frac{ih}{2\pi}\frac{d}{dt}W = -\frac{h}{8\pi m}\frac{d^2}{dx^2}W + V(x,t)W$

Order

- The order of a differential equation is the highest derivative appearing in the equation. For example, $\frac{d^3y}{dx^3} + x^2\frac{dy}{dx} + 3 = Sin(y)$ is a third order differential equation because the third derivative is the highest derivative.
- If we have an n^{th} order differential equation, we can always write it in general form: $F(x, y, y', y'', ..., y^{(n)}) = 0$, or normal form: $y^{(n)} = F(x, y, y', y'', ..., y^{(n-1)})$

Example: $x^3y'' + 3xy' + Cos(x) = y$

General form: $x^3y'' + 3xy' + Cos(x) - y = 0$

Normal Form: $y'' = \frac{y-3xy'-Cos(x)}{x^3}$

Linearity

- A differential equation is linear if in general form we can write it as $F(x, y', y'', ..., y^n) = a_1(x) + a_2(x)y + a_3(x)y' + a_4(x)y'' + ... + a_{n-1}(x)y^x$.
- When the coefficient of each y or $y^{(1)}$ is a function only of x, the differential equation is linear.

Examples:

$y'' + 3x^3y + 2 = 0$ (2^{nd} order, linear)

$y'' + y'y + 3x = 0$ (2^{nd} order, non-linear)

$y''' + y = x$ (3^{rd} order, linear)

$\frac{d^2y}{dx^2} + y^3 = x$ (2^{nd} order, non-linear)

$t^3\frac{d^2y}{dt^2} + y^7 = t^8$ (2^{nd} order, non-linear)

$(1-y)y' + 2y = e^x$ (1^{st} order, non-linear)

$y''' - Cos(x) = y$ (3rd order, linear)

$\frac{d^2y}{dx^2} - Sin(y) = xy$ (2nd order, non-linear)

$\frac{d^2y}{dx^2} + y^3 - x = 0$ (2nd order, non-linear)

Important Questions

- When does a DE:
    1. Have a solution?
    2. Have a unique solution?
    3. And where are these solutions?
- Say $\phi(x)$ is a solution on an interval I of the differential equation $F(x, y', y'', ..., y^n) = 0$ if we can substitute $y = \phi(x)$ and the equation holds for every x in I.

Example:

Consider $y'' + \frac{2}{x^2} = 0$

Check if $\phi(x) = ln(x^2)$ is a solution (If so, where?)
$\phi'(x) = \frac{1}{x^4}2x = \frac{2}{x}$
$\phi''(x) = -\frac{2}{x^2}$

Checks out, $-\frac{2}{x^2} + \frac{2}{x^2} = 0$

So $ln(x^2)$ is a solution on $(0, \infty)$ or $(-\infty, 0)$. Not $(-\infty, \infty)$