Math 374 - Spring 2020
Homework 1 - Snowplows!
Due March 9 2020

All models are wrong, some are useful.

— George Box

Turn in: (You may work in groups of 2 and submit a single assignment if you wish. Solutions must be written up carefully. A report typed up in LaTeX is recommended.)

(1) A snowplow's velocity $v(t)$ depends on the depth $D(t)$ of the snow that it is plowing. We'll approximate this relationship as

$$v(t) = k\left(\frac{1}{1+D(t)}\right)$$

where $D(t)$ is the depth of the snow at time t, and k is the maximum speed of the snowplow when there is no snow at all. Our snowplow has a maximum speed of 35 miles/hour.

Some time during the night it begins to snow at a constant rate, and continues to snow at this rate indefinitely. At 7am a snowplow begins to plow, driving in a straight line. By 8am, the snowplow has travelled 22 miles, and by 9am it has travelled 41.25 miles. Your goal is to figure out what time it started snowing.

(a) Write an equation for the depth of the snow at time t, where t measures hours after 7am. (There should be 2 unknown constants in your equation, r the rate of snowfall, and d, the depth at time $t = 0$.)

(b) Write a differential equation for the position x of the plow (in miles from the garage) at time t. (The constants k, r, and d should appear in the equation.)

(c) Solve this differential equation.

(d) Suppose that k, the top speed of the snowplow, is 35 miles/hour. Use this to solve for d and r, give their exact values to 4 decimal places. What time did it begin snowing?

Hint: You will likely need to solve an equation numerically. My suggestion is to first find the value of $\left(\frac{r}{1+d}\right)$. (To solve an equation numerically you can use the command `find_root(f(x)==0,a,b)` in SageMath to find a root of $f(x)$ in the interval $[a, b]$), and then use that value to find the other values.)

(2) A second snowplow (identical to the first) departs at 8am and follows in the first snowplow's tracks.

(a) Assume that immediately after the first snowplow passes, the depth of the snow is 0. Write a function $D_2(y, t)$ for the depth of the snow at time t and at distance y along the road. (Assuming the first snowplow has already passed.)

(b) Write a differential equation for the velocity $\frac{dy}{dt}$ of the second snowplow as a fuction of time t and position y along the road. Then use this to find a differerential equation for the change in time as a function of the position y along the road $(\frac{dt}{dy})$. (Note: this is the reciprocal of velocity! Hint: writing it this way should give a linear differential equation of t as a function of y.)

(c) Solve this differential equation.

(d) At what time does the second snowplow crash into the back of the first?