Math 369 notes

September 2025

1 Isomorphism

- if f: $G_1 \to G_2$ is a bijection and $f(gh) = f(g)f(h)$ for any two elements $g, h \in G_1$ then we call f an ismorphism and say G_1 and G_2 are isomorphic
- notation: $G_1 \cong G_2$
- Ex: $Z_6 \cong Z_2$ with isomorphism f: $Z_6 \to Z_2$ $f([1]_6) = 0$ $f([5]_6) = 1]$
- Ex $(R, +)$ and (R_+, x) are isomorphic
  - : $Z_6 \to Z_2$
  - : $f([1]_6) = 0$
  - : $f([5]_6) = 1]$
- Ex $(R, +)$ and (R_+, x) are isomorphic
  - :the isomorphism is
  - : $\phi = R \to R+$
  - : $\phi(x) = e^x = exp(x)$
  - :this is a bijection
  - : $e^x \neq e^y : fx \neq y$ so one to one
  - : Its onto since $y \in R+$ then $ln(y) \in R$ and $e^(ln(y)) = y$
  - : This is isomorphism: pick any two elements $x, y, \in R$ then need to check that $\phi(x + y) = \phi x * \phi y$ and $\phi(x + y) = e^(x + y)$ while $\phi(x)\phi(y) = e^x * e^y = e^(x + y)$
- Theorem: If G and H are isomorphic with isomorphism ϕ then
  - $$|G| = |H|$$
  - $$\phi(eq) = eH$$

  -
$$\phi(g)^{-1} = \phi(g^{-1})$$
  - G is abelian iff h is abelian

- Proof of Theorem:
  - $|G| = |H|$ since ϕ is a bijection
  -
$$\phi(a) = \phi(eg * a) = \phi(eg)\phi(a)$$
  -
$$\phi(eg)\phi(a) = \phi(ega) = \phi(a) = e_h\phi(a)$$
  -
$$\phi(eg)\phi(a) = e_h\phi(a)$$
  -
$$\phi(eg) = e_h$$
    by right cancellation
  - Claim: $\phi(g^{-1}) = \phi(g)^{-1}$
  - Proof: $\phi(g) * \phi(g^{-1}) = \phi(gg^{-}1) = \phi(e_y) = e_M$
  - So $\phi(g)\phi(g^{-1}) = e_H$
  - So $\phi(g^{-1}) = \phi(g^{-1})$ in H
  - $s^{-1} = t$

- how to check if G and H are isomorphic
  - Are Z_3 and Z_4 isomorphic?
    * No $|Z_3| = 3$ and $|Z_4| = 4$

- Are Z_4 and $< 1 > \subseteq C$ isomorphic?
  - Yes, $\phi i Z_4 \to < i >$
  - $\phi([a]_4) = i^a$

- Are Z_3 and S_3 isomorphic?
  - There are same size
  - Z_6 is cyclic and S_3 is not
  - Z_6 is abelian and S_3 is not

facts we can't prove yet

- S_3 and Z_6 are only two groups of order 6 in the sense that any group of order 6 is isomorphic to one of them

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Theorem

- If $\phi : G \to H$ is isomorphism, then order of g in 6 is equal to order of $\phi(g) in H$
- Conjugate
  - If G is a group $H \subseteq$ is a group and $a \in G$ the conjugate of H by a is
  - $$aHA^{-1} = aha^{-1}|h \in H$$
  - This is also a subgroup of G $aHA^{-1} \subseteq G$
  - Proof
  - Using 2 criteria test for a subgroup we need to show that $aHA^{(-1)}$ is non empty and $b, c \in aHA^{(-1)}$ then $bc^{(-1)} \in aHA^{(-1)}$
  - Non empty-
  - Since $e \in H$ s.t $a \in a^{-1} = aa^{-1} = e$ is in aHA^{-1}
  - Inverse
  - Now supposed $b, c \in aHA^{-1}$
  - Since $b \in aHA^{-1}$ there exist $h_1 \in H$ with $b = ah_1a^{-1}$
  - Similarly, $c = aha_2a^-1$ for some $h_2 \in H$
  - Goal: $bc^{-1} \in aHA^{-1}$
  - $bc^{-1} = (aha^{-1})(ah_2a^{-1})^{-1}$
  - $bc^{-1} = (aha^{-1})((a^-1)^{-1}h_2a^{-1})$
  - $bc^{-1} = (a(h_2)^{-1}a^{-1})$
  - $bc^{-1} = ah_1(h_2)^{-1}a^{-1}$
  - So $ah_1h_2a^{-1} \in aHa^{-1}$
  - Example: $G = S_4$
    * $G = S_4$
    * $H =< (1, 2, 3) >= 0, (1, 2, 3), (1, 3, 2)$
    * $(1, 2, 3)^2 = (1, 3, 2)$
    * Find:$(2, 4)H(2, 4)^{-1}$
    * Conjugate H by (2,4)
    * $$2, 4)H(2, 4)^{-1} = 2, 4)H(2, 4)$$
    * $$= (2, 4)e(2, 4,)^{-1}, (2, 4)(1, 2, 3)(2, 4)^{-1}, (2, 4)(1, 3, 2)(2, 4)$$

*
$$= (2,4)H(2,4) = (e),(1,4,3)(1,3,4)$$

* To find inverse you can do two line notation and do inverse

Theorem

* If G is a group and $H \subseteq H$ is a subgroup, $a \in G$ then $\phi : H \to aHa^{-1}$
* Given by: $\phi(h) = aha^{-1}$ is isomorphism