MATH 369 Fall 2025 - Class Notes

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Summary: A focus into functions and it's types, The image of functions / Well deified. Equivalence relations, partitions of sets, and factor sets.

0.1 Functions

- A Function f from a set S to a set I is a collection of ordered pairs $(s,t) \in S \times T$, where for every element $s \in S$ there is exactly one ordered pair whose first element is s.
  - **Ex**$_1$: If $S = \{0,1\}$ and $T = \{A,B,C\}$, then $f = \{(0,B),(1,C)\}$. This is a function
    (So $f(0) = B$ and $f(1) = C$)
  - **Ex**$_2$: If $S = \{0,1\}$ and $T = \{A,B,C\}$, then $f = \{(0,A),(1,A)\}$ This is also a function.
  - **Ex**$_3$: If $S = \{0,1\}$ and $T = \{A,B,C\}$, then $f = \{(0,A),(1,C)(0,B)\}$ This is **not** a function (Same input, two different Output).
- A function needs to be well defined, meaning one input should have only one output. Here are some examples that aren't true.
  - **Ex**$_4$ $f : \mathbb{Q} \longrightarrow \mathbb{Z}$. f is $f(\frac{n}{m}) = m$, We put in a fraction and it's denomator is our output. Take the input $\frac{1}{2}$. $\frac{1}{2} = \frac{2}{4}$, but $f(\frac{1}{2}) = 2$ and $f(\frac{2}{4}) = 4$, so f is not a well defined fuction.
  - **Ex**$_5$ $f : \mathbb{R} \longrightarrow \{0,1,...,9\}$. where $f(d_1, d_2, ..., d_k.d_i...) = d_i$. (This function takes in any number in base 10 and grabs the first digit after the decimal point. for

example, $f(\pi) = 1$.)

The issue with this function is that since $1 = 0.\overline{9}$, but $f(1) = 0$ and $f(0.\overline{9}) = 9$

- When looking at the set S in our definition for function, S is called our domain (All of our inputs). T is called our co-domain (all of the possible outputs). While the set $\{f(s)|s \in S\} \subset T$ is called the image (The image is all of the outputs of our function).

  Looking at earlier examples, **Ex**$_1$ has an Image of B and C, written as $Im(f) = \{B, C\}$. Let's look at one more example:

  - **Ex**$_6$: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$. f's domain is $(-\infty, \infty)$, it's co-domain is the same, but f's image is $[0, \infty)$.

- A function f is considered injective (One-to-one), if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

  - **Ex**$_7$: Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$. $f(x) = x^2$ is not an injective function. Since $f(-2) = 4$ and $f(2) = 4$ but $-2 \neq 2$, f is not an injective function.

- A function f is considered surjective (onto), if for each element $t \in T$, there exists $s \in S$ such that $f(s) = t$ (Which means, every element in the co-domain [T], their is an input that gets there. If a function is onto, than its co-domain should be equal to the image.)

  - **Ex**$_8$: Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$. $f(x) = x^2$ is not a surjective function. If $x > 0$ or $x < 0$, then $x^2 > 0$. If $x = 0$, then $x^2 = 0$. Therefore, the image of f is $[0, \infty)$, leaving elements in our co-domain $(-\infty, -1]$ not able to be reached. So f is not surjective.

- If f is both surjective and injective, then the function f is a bijection.

- Often if we have a function $f : S \longrightarrow T$, we will want to look at the inverse function, $f^{-1} : T \longrightarrow S$ (The function should operate backwards, so if $f(s_1) = t_1$, then $f^{-1}(t_1) = s_1$). A function can only have an inverse function if it's a bijective function (if f isn't surjective, such as in**Ex**$_8$, numbers such as -2 won't be able to be put in the function.

if f isn't injective, such as in **Ex**$_7$, if I input 4 then I could get -2 or 2 which makes this not a function.).

0.2 Relations

- We write $\sim$ for a Relation on a set A. A relation is a subset of ordered pairs $(a, b) \in A \times A$. A pair (a, b) in our relation is written as $a \sim b$.
- There are **3** properties that are necessary for an Equivalence Relation.
  - Reflexive: $a \sim a$ for all $a \in A$ (a element in A must relate to itself.)
  - Symmetric: If $a \sim b$ then $b \sim a$ for all $a, b \in A$. (It must work both ways! If a relation uses $(a, b) \sim (c, d)$, then $(c, d) \sim (a, b)$ must also work).
  - Transitive: If $a \sim b$ and $b \sim c$, then $a \sim c$ for all $a, b, c \in A$. (The order matters here! Make sure it's " If 1 than 2, and if 2 than 3, then if 1 then 3".)
- **Ex**$_9$: We'll look at congruence with (mod n). Our Relation is on $\mathbb{Z}$, So $a \equiv b \mod n$ iff $n \mid (b - a)$.
  - Reflexive: Is $a \equiv a \mod n$? Yes! Since $a - a = 0$, $n \mid 0 = 0$
  - Symmetric: If $a \equiv b \mod n$, then does $b \equiv a \mod n$? Yes! If $n \mid (b - a)$, then $n \mid (a - b)$ or rather $n \mid -(b - a)$.
  - Transitive: If $a \equiv b \mod n$ and $b \equiv c \mod n$, does $a \equiv c \mod n$? Yes! We know $n \mid (b - a)$ and $n \mid (c - b)$, and since $(b - a) + (c - b) = c - a$, then $n \mid (c - a)$
- A partition of a set S is a collection of disjoint subsets of S such that every element of S is contained in exactly one subset. (Using Dr. McNew's analogy, If we put all of S's elements in multiple buckets, one number can only be in one bucket and **not** strecthed between two).
- If $\sim$ is an Equivalence Relation, we define the Factor Set (written as $S/\sim$) as the partion of our set from the equivalence relation (Given by $\sim$).

Theorems and Proofs:

Theorem 1. *If $I \subset \mathbb{Z}$ that is closed under addition and subtraction, then either*

- *(a) I is empty*
- *(b) $I = \{\emptyset\}$*
- *(c) $I = a\mathbb{Z} = \{ak \mid k \in \mathbb{Z}\}$ for some a.*

Proof. If I is one of the sets listed above, then (a) is clear, (b) 0+0=0=0-0 (Closed), (c) is proved from the Propisition earlier in class.
Suppose I is closed under addition and subtraction. If I is empty or $I = \{0\}$, we are done. (We want an object to go and play with, so by eliminating these options we can work on the proof.) Otherwise, I contains atleast one non-zero element, call it b. If $b < 0$, then $b - b - b > 0$ (This would make it positive.) This number would have to be in the set, since I is closed under addition and subtraction. Relabel if nessacery this element to be b.
Because $I \subseteq \mathbb{Z}$ and it contains some positive number, I contains a least posiitve number, call that a, where $a \in I$. Since I is closed under addition and subtraction, $a\mathbb{Z} \in I$. (The easiest way to prove this would be by contradiction, what is what we'll do.)
We need to show that $a\mathbb{Z} = I$. Suppose $c \in I$ but $c \notin a\mathbb{Z}$. Since $a \nmid c$ (c isn't a multiple of a since $c \notin a\mathbb{Z}$, so it doesn't divide it evenly.), We can use the **division algorithm** on a and c. It tells us that $c = aq + r$ with $r \neq 0$, but $a > r > 0$ by the Division Algorithm. So $r = c - qa \in I$ (Because I is closed under addition and subtraction.) So r is a smaller, posiitive element in I than a, a contradition. SO $I = a\mathbb{Z}$ $\square$

Theorem 2. *"Set partitions and Equivilance Relations are the same thing" — If $\sim$ is an Equivilance Realtion then the Equivilance Classes (The sets of all thngs that are equivilant) form a set partion.* ***OR*** *If $P_1, P_2, ..., P_k$ is a set partion of a set S (Imagine $P_1, P_2, ..., P_k$ as the buckets S is split into) then $\sim$ is defined by $a \sim b$ iff $a \in P_i$ and $b \in P_i$ for the same index i (Same bucket) is an equivalence relation.*