MATH 369 Fall 2025 - Class Notes

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Summary: An Introduction to the class. We went over the syllabus and how class work will be. Review of Topics 1.1-1.4

Notes: Useful things about **integers**:

- Useful way to Represent them
- Every integer has a unique number that's larger and smaller than it.
- Some subsets are easy to identify (Even, Odd, Prime)
- Addition, Subtraction, and Multiplication have closure
- Useful to "count" things in the real world

Operations using Integers,

- **Addition:** Commutative, Associative ($a+b=b+a$), has an Identity (0) $\longrightarrow a+0=a$, Closed ($a+b \in \mathbb{Z}$)
- **Subtraction:** Closed ($a-b \in \mathbb{Z}$), Creates a need for "negatives", Has a one sided identiy ($a-0=a$ but $0-a \neq a$)
- **Multiplication:** Commutative, Associative, Closed (within integers), has an identity (1).
- **Division:** Not always Defined (0), not Commutative, not closed ($\frac{2}{3}$ does not simplify into an integer), not Associative, has a one sided identity ($\frac{a}{1}=a$ but $\frac{1}{a} \neq a$).

We say that a **divides** b if $a, b \in \mathbb{Z}$ and their exists $k \in \mathbb{Z}$ such that $b = ka$. We write this as $a \mid b$.

Proposition: If $a \mid b$ and $a \mid c$, then $a \mid (bm + cn)$ for any integer m and n.

Proof. Since $a \mid b$, their exists a k such that $ak = b$. Since $a \mid c$, their exists a l such that $ak = l$. We substitute $a \mid (bm + cn) = a \mid ((ak)m + (al)n)$. By distributive property, $a \mid ((ak)m + (al)n) = a \mid a(km + ln)$. Since $a \mid a$, $a \mid (bm + cn)$ by definition. $\square$

Modular Arithimitic: Say that a is congruent to b modulo n and write $a \equiv b \mod n$. If n divides $(b - a)$, a and b have the same remainder when divided by n.

Theorems and Proofs: Use theorem environments for important results:

Theorem 1. ***Division Algorithm****: If $a, b \in \mathbb{Z}$ with $b > 0$, then there exists unquie q and r where $q, r \in \mathbb{Z}$ with $0 \leq r < q$, such that $a = bq + r$, where $q = \lfloor \frac{a}{b} \rfloor$, and $r = a - bq$.*

Theorem 2. *If $I \subset \mathbb{Z}$ that is closed under addition and subtraction, then either*

- *(a) I is empty*
- *(b) $I = \{\emptyset\}$*
- *(c) $I = a\mathbb{Z} = \{ak \mid k \in \mathbb{Z}\}$ for some a.*

Proof. If I is one of the sets listed above, then (a) is clear, (b) 0+0=0=0-0 (Closed), (c) is proved from the Propisition earlier in class.
Suppose I is closed under addition and subtraction. If I is empty or $I = \{0\}$, we are done. (We want an object to go and play with, so by eliminating these options we can work on the proof.) Otherwise, I contains atleast one non-zero element, call it b. If $b < 0$, then $b - b - b > 0$ (This would make it positive.) This number would have to be in the set, since I is closed under addition and subtraction. Relabel if nessacery this element to be b.
Because $I \subseteq \mathbb{Z}$ and it contains some positive number, I contains a least posiitve number, call that a, where $a \in I$. Since I is closed under addition and subtraction, $a\mathbb{Z} \in I$. (The easiest way to prove this would be by contradiction, what is what we'll do.)

We need to show that $a\mathbb{Z} = I$. Suppose $c \in I$ but $c \notin a\mathbb{Z}$. Since $a \nmid c$ (c isn't a multiple of a since $c \notin a\mathbb{Z}$, so it doesn't divide it evenly.), We can use the **division algorithm** on a and c. It tells us that $c = aq + r$ with $r \neq 0$, but $a > r > 0$ by the Division Algorithm. So $r = c - qa \in I$ (Because I is closed under addition and subtraction.) So r is a smaller, posiitive element in I than a, a contradition. SO $I = a\mathbb{Z}$ $\square$