MATH 369: SECTION 5.4 QUOTIENT FIELDS
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Class Summary

In this section we construct the quotient field (also called the field of fractions) of an integral domain. Just as $\mathbb{Q}$ is formed from $\mathbb{Z}$ by adjoining multiplicative inverses of all nonzero elements, every integral domain D naturally embeds in a field built from "fractions" of elements of D.

This construction is essential because many algebraic techniques require a field, and integral domains may not have multiplicative inverses.

0.1. Motivation

Given an integral domain D, we want to create a field F in which:

1. D is a subring of F,
2. every nonzero element of D has an inverse in F,
3. F is "smallest" with these properties.

The analogy is:

$$\mathbb{Z} \subset \mathbb{Q}, \qquad F[x] \subset F(x).$$

0.2. Construction of the Quotient Field

Let D be an integral domain. Consider ordered pairs (a, b) where $a, b \in D$ and $b \neq 0$. Define the equivalence relation:

$$(a, b) \sim (c, d) \iff ad = bc.$$

The equivalence class of (a, b) is written $\frac{a}{b}$.

Operations

For $\frac{a}{b}$ and $\frac{c}{d}$ in the quotient field:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd},$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Theorem. With these operations, the set of equivalence classes forms a field.
We call this field the quotient field of D, denoted $\mathrm{Quot}(D)$ or sometimes $\mathrm{Frac}(D)$.

0.3. Canonical Embedding of D into $\mathrm{Quot}(D)$

Define

$$\iota : D \longrightarrow \mathrm{Quot}(D), \qquad \iota(a) = \frac{a}{1}.$$

This map is injective, so we identify D with a subring of its quotient field.

0.4. Universal Property of the Quotient Field

Theorem. Let D be an integral domain and let F be a field containing a subring isomorphic to D. Then there exists a unique injective ring homomorphism

$$\phi : \mathrm{Quot}(D) \to F$$

extending the embedding of D into F.

Interpretation: The quotient field is the "smallest" field that contains D.

0.5. Examples

Example: $\mathbb{Z}$

The quotient field of $\mathbb{Z}$ is $\mathbb{Q}$. Fractions are of the form $\frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b \neq 0$.

Example: $F[x]$

For a field F, the integral domain $F[x]$ has quotient field

$$F(x) = \left\{ \frac{f(x)}{g(x)} : f, g \in F[x],\ g \neq 0 \right\},$$

the field of rational functions.

Example:

$$\frac{x^3 - 2x}{x^2 + 1} \in F(x),$$

and

$$\frac{x^3 - 2x}{x^2 + 1} + \frac{3x - 1}{x - 2} = \frac{(x^3 - 2x)(x - 2) + (3x - 1)(x^2 + 1)}{(x^2 + 1)(x - 2)}.$$

Example: $\mathbb{Z}[\sqrt{2}]$

Let

$$D = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}.$$

Its quotient field is

$$\mathrm{Quot}(D) = \mathbb{Q}(\sqrt{2}) = \left\{ \frac{a + b\sqrt{2}}{c + d\sqrt{2}} : c + d\sqrt{2} \neq 0 \right\}.$$

Example: Rationalizing the denominator:

$$\frac{1}{3+\sqrt{2}} = \frac{3-\sqrt{2}}{(3+\sqrt{2})(3-\sqrt{2})} = \frac{3-\sqrt{2}}{7}.$$

Example: Non-UFD Domain

Let

$$D = \mathbb{Z}[\sqrt{-5}].$$

The quotient field is

$$\mathbb{Q}(\sqrt{-5}),$$

even though D is not a UFD. The quotient field exists *whenever* D is an integral domain.

0.6. Embedding of Rational Functions

Any field F embeds into its rational function field:

$$F \hookrightarrow F(x), \qquad a \mapsto \frac{a}{1}.$$

Example:

The element

$$\frac{x^4 + 3x^2 - 1}{x^3 - 2}$$

is in $F(x)$ but not in $F[x]$.

0.7. Summary

- Every integral domain D has a quotient field $\mathrm{Quot}(D)$.
- Elements are equivalence classes of pairs (a, b) with $b \neq 0$.
- D embeds naturally into $\mathrm{Quot}(D)$ via $a \mapsto \frac{a}{1}$.
- The quotient field is minimal with the property that every nonzero element of D becomes invertible.
- Examples include:

$$\mathbb{Z} \subset \mathbb{Q}, \quad F[x] \subset F(x), \quad \mathbb{Z}[\sqrt{d}] \subset \mathbb{Q}(\sqrt{d}).$$