**MATH 369: SECTION 5.3 IDEALS AND FACTOR RINGS**
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# Class Summary

In this section we introduce the concept of ideals, which play the same structural role in ring theory that normal subgroups play in group theory. Ideals are the subsets that allow us to construct well-defined quotient rings. We also connect ideals to kernels of ring homomorphisms and explore examples in $\mathbb{Z}$ and polynomial rings.

# Definition of Ideals

Let $R$ be a ring. A subset $I \subseteq R$ is called an ideal if:

1. $I$ is an additive subgroup of $R$.
2. For all $r \in R$ and $a \in I$, we have $ra \in I$ and $ar \in I$. (In commutative rings, only $ra \in I$ is required.)

**Motivation:** Ideals are exactly the subsets for which the quotient $R/I$ becomes a ring with well-defined multiplication.

# Examples of Ideals

**Example 1:** Every ideal in $\mathbb{Z}$ has the form $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$.
**Example 2:** In $F[x]$, an ideal generated by a polynomial $f(x)$ is

$$(f(x)) = \{f(x)g(x) : g(x) \in F[x]\}.$$

**Example 3:** The sets $\{0\}$ and $R$ are ideals in any ring $R$ (called improper ideals).

# Principal Ideals

If $a \in R$, the set

$$(a) = \{ra : r \in R\}$$

is the principal ideal generated by $a$.

**Examples:**

- In $\mathbb{Z}$ every ideal is principal.
- In $F[x]$, every ideal is principal since $F[x]$ is a PID.

# Factor Rings

If $I$ is an ideal of $R$, the factor ring $R/I$ is the set of cosets:

$$R/I = \{r + I : r \in R\}.$$

**Operations:**

$$(r + I) + (s + I) = (r + s) + I$$
$$(r + I)(s + I) = rs + I$$

**Theorem.** If $I$ is an ideal of $R$, then $R/I$ is a ring.

# Ideal Test

A subset $I \subseteq R$ is an ideal if:

1. $I$ is closed under addition and additive inverses.
2. For all $r \in R$ and $a \in I$, $ra \in I$.

**Example.** $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$. **Non-example.** $3\mathbb{Z} + 1$ is not an ideal.

# Kernels of Ring Homomorphisms

If $\phi : R \to S$ is a ring homomorphism, then

$$\ker(\phi) = \{r \in R : \phi(r) = 0\}$$

is always an ideal of $R$.

**Theorem (Ring Homomorphism Theorem).** If $\phi : R \to S$ is a surjective ring homomorphism, then

$$R/\ker(\phi) \cong \mathrm{Im}(\phi).$$

# Examples of Factor Rings

**Example 1:** $\mathbb{Z}/n\mathbb{Z}$ is the ring of integers mod $n$.
**Example 2:** $F[x]/(f(x))$ consists of equivalence classes of polynomials modulo $f(x)$. If $f(x)$ is irreducible, then $F[x]/(f(x))$ is a field.
**Example 3:** $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{C}$.

# Characterization of Ideals in $\mathbb{Z}$

**Theorem.** Every ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some integer $n \geq 0$.

**Proof.** If $I$ is a nonzero ideal, let $n$ be the smallest positive integer in $I$. Show $I = n\mathbb{Z}$ using the division algorithm.

# Prime and Maximal Ideals

**Prime ideal:** An ideal $I \subsetneq R$ is prime if

$$ab \in I \Rightarrow a \in I \text{ or } b \in I.$$

**Maximal ideal:** $I \subsetneq R$ is maximal if no ideal lies strictly between $I$ and $R$.
**Theorem.** For a commutative ring $R$:

- $R/I$ is an integral domain $\iff$ $I$ is prime.
- $R/I$ is a field $\iff$ $I$ is maximal.

# Summary

- Ideals generalize normal subgroups.
- Factor rings $R/I$ behave like quotient groups but with multiplication defined.
- Kernels of homomorphisms are always ideals.
- In $\mathbb{Z}$ and $F[x]$, every ideal is principal.
- Factor rings give useful constructions like $\mathbb{Z}/n\mathbb{Z}$ and fields $F[x]/(f(x))$.