MATH 369: SECTION: 4.3 POLYNOMIALS OVER Q, R, AND C 11/05/2025

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Class Summary

We started the class by focusing onto the ways to test if Polynomials with integer coefficients are irreducible over the field of rational numbers or have rational roots or not. We focused mostly on the definition of Eisenstein's Irreducibility Criterion.

0.1. **Ways to Test Polynomials are irreducible.** When does f(x) $\in Q[x]$ have a root? Suppose f(x)=$a_n x^n$+$a_{n-1} * x^{n-1} + - - - - + a_0$ and s/t is a root of f(x) in lowest form(so gcd(s,T)=1). Then s/a_0 and T/a_n.

0.2. **Proof.**

Proof. If s/T is a root then 0=f(s/T) $= a_n(s/T)^n + a_{n-1}(s/T)^{n-1} + - - - - + a_1(s/T) + a_0$ then multiplying both sides by Tn we got 0=$a_n s^n + a_{n-1}s^{n-1} - - - - - a_1 s$ (which is divisible by s and the last term $a_0 t^n$ which is divisible by s and except the first term everything is divisible by T and that is how we get s/a_0 and T/a_n

□

0.3. **Example 1.** $f(x) = 2x^3 - 3x^2 - 11x + 6$ Suppose s/T is a factor then s/6 so s $\in (\pm 1, \pm 2, \pm 3, \pm 6)$ also T/2 so t$\in$($\pm 1, \pm 2$)
Taking T= 1,s=1 then f(1)=2-3-11+6 = -6
Taking T=1,s=2 then f(2)=16-12-22+6 = -12
when T=1,s=3, then f(3)= 0 so we need to divide f(x) by (x-3) and when we divide it we get quotient as g(x)= $2x^2 + 3x - 2$ and remiander, 0 so f(x)=(x-3)($2x^2 + 3x - 2$) and after that the possible roots=(1,2,-1,-2,1/2,-1/2) and when we substitude -2 in g(x) we get g(-2)=0 and that means the 2 is also a root. // Again when we divide g(x) by x+2 we get quotient 2x-1 and remainder, o so our f(x)=(x-3)(x+2)(2x-1) and that how we get our roots; 3,-2,1/2 //We can also use this exact same method to prove if things are irreducible of the degree is 3 or smaller.

0.4. **Example 2.** EX: $x^2 - 3$ rational root theorem says the possible roots are($\pm 1, \pm 3$) //None of ± 1, ± 3 are roots this polynomial has no rational roots so it is irreducible.

0.5. **Definition:Primitive.** A polynomial in Z[x] is primitive if the gcd of the coefficients is 1. We call the gcd of the coefficients the content of the polynomial so a primitive polynomial is a polynomial with content 1.
Example
f(x)=$3x^2 - 12x + 9$ is not primitive because it has content 3.

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0.6. **Lemma.** If f(x)=$a_nx^n + a_0$ and g(x)= $b_mx^m + b_0$
and p is a prime.// Let i be the least index so p is not equal to a_i
let j be the least index so p is not equal to b_j//then i+ j is the lease index so that p doesn't divide the coefficient of x^{i+j} in f(x)g(x).
Proof
Multiply out f(x)g(x), find the coefficient of x^{i+j}
Since the coefficient is the sum of a bunch of things divisible by p and a_ib_j which isn't this coefficient isn't divisible by p.
Ex: f(x)=$x^2 + 2x + 2$
g(x)=$x^3 - 5x + 4$,
p=2
i=2
j=1
f(x)g(x) then the least coefficient not divisible by 2 is x^3.
$(x^2 + 2x + 2)(x^3 - 5x + 4)$———+(-5+2)$x^3$+(0-0+4)$x^2$+ (8-10)$x^1$+ 2(4)$x^0$-3

0.7. **Gauss' Lemma.** If f(x) and g(x) are permutative then so is f(x)g(x)
Proof:
Suppose f(x)g(x) weren't primitive for a contradiction. let p be a prime that divides the content of f(x)g(x) since f and g are primitive there coefficients arent all divisible by p. let r and j be the least indices of coeffcients not divisible by p in each
By the previous lemma p does not divide the coefficient of x^{i+j} in f(x)g(x)
This contradicts p dividing the content of f(x)g(x)
So f(x)g(x) is also primitive
Theorem :
If f(x) $\in$ Z[x] and it factors in Q[x] then it factors in Z[x].
Any polynomial with integer coefficents that factors in Q[x] actually factors into polynomials with integer coefficients.
Theorem: Eisensteins Criterion
Suppose f(x)=$a_nx^n + a_{n-1}x^{n-1} - - - - - a_0$and p is a prime.
If a_n doesn't divide p
p$/a_{n-1}, p/a_{n-2} - - - - - - - - p/a_0$
a_0 doesn't divide p^2 then f(x) is irreducible
Proof
Suppose fpr a contradiction that such as f(x) factors as f(x)=b(x)c(x)
b(x)=$b_jx^ic + b_0$
c(x)=$c_jx^ic + c_0$
consider the constant term of bothb_0 and c_0
$a_0 = b_0c_0$
since a_0 doesn't divide p^2, p can't divide both$_b$0 and c_0 // so b_0 isn't divisible by p
Example:
f(x)=$x^n - 2$ for any $n > 1$ is irreducible.
pick p=2// p divides evey coefficent except the leading coefficient, it doesn't divide 1 so by Eisenstein this is irreducible

Example:
f(x)=$2x^3 + 3x^2 - 9x - 6$
pick p=3
3 divided 3,-9,6 but it doesn't divide 2 and 3^2 isn't divisible by -6 so this is irreducible by Eisenstein.
TRICK

Sometimes we have a polynomial like f(x)=$x^4 - 3x^3 + 2x + 1$– No prime factors for Eisensteins criterias,
suppose f(x)=g(x)h(x)
consider f(x+a)=g(x+c)h(x≠a)
f(x) is irreducible if and only if f(x+a)is
consider f(x)=$x^4 - 3x^3 + 2x + 1$
f(x+1)=$(x+1)^4$-3$(x+1)^3$+2(x+1)+1=$x^4 + 3x^3 + 3x^2 + 3x + 3$
which is irreducible by Eisenstein