MATH 369 Fall 2025 - Class Notes

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Summary: We discussed homomorphisms, kernels, and equivalent classes.

Notes:

If $\phi : H \to G$ is a homomorphism and $\text{Ker}(\phi) = \{h \in H | \phi(h) = e_G\}$, then $\text{Ker}(\phi)$ is a subgroup of H.
H/ϕ is also a group.

Reminder:
H/ϕ is the set of all equivalence classes under the equivalence $a \sim b$ if $\phi(a) = \phi(b)$.

Example
$\phi : \mathbb{Z} \to \mathbb{Z}_3$
$\phi(a) = [a]_3$
The equivalence classes of $\mathbb{Z}/\phi$ are

- $\{0, 3, 6, 9, ...\}$
- $\{1, 4, 7, ...\}$
- $\{2, 5, 8, ...\}$

Theorem 1. *If $\phi : H \to G$ is a homomorphism, then H/ϕ is a group with operation composition of equivalence classes.*

Claim:
$[a]_\phi * [b]_\phi = [ab]_\phi$ is well-defined.

If we replace a, b with different representations of the same class, do we get the same answer?

Suppose that
$c \sim_\phi a$, $d \sim_\phi b$

We need to show that $[a]_\phi * [b]_\phi = [ab]_\phi$ is the same as $[cd]_\phi$.

Since $c \sim_\phi a$ and $d \sim_\phi b \to \phi(c) = \phi(a)$, $\phi(d) = \phi(b)$
So

$$\phi(ab) = \phi(a)\phi(b) = \phi(c)\phi(d) = \phi(cd)$$

Then

$$[a]_\phi [b]_\phi = [ab]_\phi = [cd]_\phi$$

so this operation is well-defined.
Now check this satisfies the group properties.

- Identity: $[e_H] = \text{Ker}\phi$ is the identity.
- Associative: Since elements of H are associative.
- Closed Under Composition: By construction.
- Inverses: $[a]_\phi^{-1} = [b]_\phi^{-1}$

Example: $\phi : \mathbb{Z}_6 \to \mathbb{Z}_{10}$
$\phi([a]_6) = [5a]_{10}$

Why is this well-defined???

$\phi([a]_6) = [7a]_{10}$ is not well-defined since $[2]_6 = [8]_6$, but $[14]_{10} \neq [56]_{10}$.

We will prove that this is well defined:

Proof. Suppose

$$a \equiv b (mod 6)$$

then

$$b - a \equiv 0 (mod 6)$$

so

$$(b - a) = 6k$$

so

$$b = a + 6k$$

$$\phi(b) = [5b]_{10} = [5(a + 6k)]_{10} = [5a + 30k]_{10}$$

Since $[30k]_{10} = [0]_{10}$, we see that $[5b]_{10} = [5a]_{10}$.
So ϕ well-defined. □

ϕ is a homomorphism since

$$\phi([a]_6 + [b]_6) = \phi([a + b]_6) = [5a + 5b]_{10} = [5a]_{10} + [5b]_{10} = \phi([a]_6) + \phi([b]_6)$$

$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

$$\phi(0) = 0$$

$$\phi(1) = 5$$
$$\phi(2) = 10 \equiv 0(mod 10)$$
$$\phi(3) = 15 \equiv 5(mod 10)$$
$$\phi(4) = 20 \equiv 0(mod 10)$$
$$\phi(5) = 25 \equiv 5(mod 10)$$

$\mathbb{Z}_6/\phi$ is a group.

What are the equivalence classes?
$[0]_\phi = \{0, 2, 4\}$
$[5]_\phi = \{1, 3, 5\} = [3]_\phi$

Here is the Composition table of $\mathbb{Z}_6/\phi \cong \mathbb{Z}_2$:

	$[0]_\phi$	$[3]_\phi$
$[0]_\phi$	$[0]_\phi$	$[3]_\phi$
$[3]_\phi$	$[3]_\phi$	$[0]_\phi$

In $\mathbb{Z}_{10}$, we find that these elements map to $\{0, 5\} =< 5 >\subseteq Z_{10} \cong \mathbb{Z}_2$.

<u>Recall</u>: If f is any function, then we can decompose f as

$$\begin{tikzcd} s \arrow[r, "f"] \arrow[d, "\pi"'] & T \\ s/f \arrow[r, "\bar{f}"] & \mathrm{Im}(f) \arrow[u, "i"'] \end{tikzcd}$$

$f = i \circ f \circ \pi$

Theorem 2. *If $f : G \to H$ is a homomorphism then $\bar{f} : G/f \to Im(f)$ is an isomorphism.*

So $G/f \cong \mathrm{Im}(f)$

<u>Example</u>:

$$\phi : \mathbb{Z}_6 \to \mathbb{Z}_{10}$$
$$\phi([a]_6) = [5a]_{10}$$

then

$$\mathbb{Z}_6/\phi \cong \mathrm{Im}(\phi)$$

where $\mathbb{Z}_6/\phi = [0]_\phi, [3]_\phi = \{0,2,4\}, \{1,3,5\}$ and $\mathrm{Im}(\phi) = \{[0]_{10}, [5]_{10}\}$

Note if G is a cyclic group with generator g, $G =< g >$ then any homomorphism $f : G \to H$ is determined by $f(g)$.

What are the homomorphisms from $\mathbb{Z}_4 \to V_4$?
All we need to do is check where $[1]_4$ goes.

<u>Case 1</u>:

$$f([1]_4) = (0,1), f([2]_4) = f([1]+[1]) = f([1]) + f([1]) = (0,1) + (0,1) = (0,0)$$

$f([3]_4) = (0,1)$
$f([0]) = (0,0)$
$\mathrm{Ker}(f) = 0, 2 =< 2 >\subseteq Z_4$

$$\mathbb{Z}_4/f = \left\{ \begin{array}{c} \mathrm{Ker} f \\ \{1,\ 3\} \end{array} \right\} \cong \mathrm{Im}(f) = \left\{ \begin{array}{c} (0,0) \\ (0,1) \end{array} \right\}$$

<u>Case 2</u>:

$$g : \mathbb{Z}_4 \to V_4 \cong \mathbb{Z}_2 * \mathbb{Z}_2$$

- $g(1) = (1,1)$
- $g(2) = (0,0)$
- $g(3) = (1,1)$
- $g(0) = (0,0)$

$\mathrm{Ker}(g) = \{0,2\}$

$$\mathbb{Z}_{f/g} = \left\{ \begin{array}{c} \mathrm{Ker}(g) = \{0,2\} \\ [1] = \{1,3\} \end{array} \right\} \cong \left\{ \begin{array}{c} (0,0) \\ (1,1) \end{array} \right\}$$

What if we are mapping $\mathbb{Z}_4 \to \mathbb{Z}_6$?
Where can $1 \in \mathbb{Z}_4$ go?
$f([1]_4) \to [1]_6$ doesn't work.

What happens if we consider $[0]_4 = [4]_4 = [1]_4 + [1]_4 + [1]_4 + [1]_4$?
$f([0]_4) = [0]_6$

$$f([4]_4) = f([1]_4 + [1]_4 + [1]_4 + [1]_4) = [1]_6 + [1]_6 + [1]_6 + [1]_6 + [1]_6 + [1]_6$$

so it is not well-defined.

The only places $[1]_4$ can go in $\mathbb{Z}_6$ are $[0]_6$ and $[3]_6$.

$$\text{Identity} = f(4[1]_4) = 4f([1]_4) = \text{Identity}$$

If $G =< g >$ is a cyclic group, a homomorphism from G must send a generator g to an element with order dividing $|G|$

What homomorphisms can we find from $\mathbb{Z}_{12} \to \mathbb{Z}_{18}$?
The homomorphism is determined by where we send $[1]_{12}$.

- $[1]_{12} \to [0]_{18}$ (order 1)
- $[1]_{12} \to [9]_{18}$ (order 2)
- $[1]_{12} \to [6]_{18}$ (order 3)
- $[1]_{12} \to [12]_{18}$ (order 3)
- $[1]_{12} \to [3]_{18}$ (order 6)
- $[1]_{12} \to [15]_{18}$ (order 6)

$a(mod n)$ has order $n/\gcd(a, n)$.

What are the homomorphisms from $D_4 \to \mathbb{Z}_4$?
$s =$ rotation of a square by $90°$
$r =$ reflection across vertical line

$D_4 = \{e, s, s^2, s^3, r, rs, rs^2, rs^3\}$
$rs \neq sr$
$rs = s^3r = s^{-1}r$.