MATH 369 Fall 2025 - Class Notes

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Summary: We discussed the end of **Chapter 3.8**, which included discussion about normal subgroups, cosets, the center of a group, $Z(G)$, properties of normal subgroups, and different ways to figure out if a subgroup is normal.

- **Definition.** Let $H \subseteq G$ be a subgroup of G. The index of a subgroup is the number of (left or right) cosets of H in G, denoted by $[G : H]$.
- **Proposition.** In a group G with a subgroup H, every coset of H has the same size as H. So if G is finite, then $[G : H] = \frac{|G|}{|H|}$.

Theorem 1. *If $[G : H] = 2$, then H is normal in G.*

Proof. H has two cosets in G. One is H itself, the other is G/H. Since these are the only cosets, the left coset gH is either H or G/H. If $gH = H$, then $g \in H$. So $gh = H = Hg$. If $gH = G/H$, then $g \notin H$. So $gH = G/H = Hg$. In either case, $gH = Hg$, so H is normal in G. □

Theorem 2. *If G is abelian and $H \subseteq G$ is a subgroup then H is normal in G.*

Proof. Not given □

- If H $\trianglelefteq$ G, then we know that G/H is a group consisting of all cosets of H in G. $(aH)(bH) = (ab)H$.
- Note that G/H is not a subgroup of G.
- $\pi : G \to G/H$, $\pi(a) = aH$ is a homomorphism called natural projection.
- $ker(\pi) = \{a \in G : \pi(a) = H\} = H$.
- Supposed $H \trianglelefteq G$, is there some homomorphism whose kernel is H? Yes. H is always the kernel of the natural projection $\pi : G \to G/H$.
- **Corollary.** Subgroups are normal if and only if they are kernels of some homomorphism.
- **Example.** Let $G = \mathbb{Z}_{20}$, $H = \langle 4 \rangle = \{0, 4, 8, 12, 16\}$. H is normal because G is abelian.
  $0 + \langle 4 \rangle = \langle 4 \rangle$.
  $1 + \langle 4 \rangle = \{1, 5, 9, 13, 17\}$.
  $2 + \langle 4 \rangle = \{2, 6, 10, 14, 18\}$.
  $3 + \langle 4 \rangle = \{3, 7, 11, 15, 19\}$.
  4 cosets, $[\mathbb{Z}_{20} : \langle 4 \rangle] = \frac{|G|}{\langle 4 \rangle}$. $\mathbb{Z}_{20}/\langle 4 \rangle \cong \mathbb{Z}_4$. $f(a + \langle 4 \rangle) = [a]_4$ (isomorphism).

- Various ways to show that $H \trianglelefteq G$:

1. Show that $gH = Hg, \forall g \in G$.
2. Find a homomorphism $\phi : G \to G_2$ with $ker(\phi) = H$.
3. Show that $gHg^{-1} \subseteq H, \forall g \in G$.

- The center of a group $Z(G)$ is the set of all elements of G that commute with the entire group.
- $Z(G) = \{z \in G : zg = gz, \forall g \in G\}$
- $Z(G)$ is always a subgroup.
- **HW Problem.** $Z(G) \trianglelefteq G$. Assume we know this.

Theorem 3. *Prove that if $G \backslash Z(G)$ is cyclic, then G is abelian.*

Proof. Since $G/Z(G)$ is cyclic, it has a generator. For some $g \in G$, $gZ(G)$ is a generator. $G/Z(G) = \langle gZ(G) \rangle$. Then every coset in $G/Z(G)$ looks like $(gZ(G))^k = g^k Z(G)$. Let $a, b \in G$. Then $a \in g^i Z(G)$ and $b \in g^j Z(G)$ for some i, j. $a = g^i z$ for $z \in Z(G)$ and $b = g^j z'$ for $z' \in Z(G)$. $ab = (g^i z)(g^j z') = g^{i+j} z' z$, $ba = (g^j z')(g^i z) = g^{j+i} z' z$. Since $z, z' \in Z(G)$ commute, and addition is commutative, $ab = ba$. Thus G is abelian. $\square$