MATH 369 Fall 2025 - Class Notes

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Scribe: Tessa Scheenstra

Summary: We talked about normal subgroups, quotient groups and cosets.

Subgroups and Cosets

Let $H \subseteq S_n$. Let $A \subseteq H$ where

$$A = \{h \in H \mid h \text{ is even}\}, \quad B = \{h \in H \mid h \text{ is odd}\}.$$

Then $H = A \cup B$ and $A \cap B = \emptyset$.

Let g be an odd element of H. If $h \in H$ is even, then gh is odd, and if $h \in H$ is odd, then gh is even. Thus, the map

$$\phi : B \to A, \quad \phi(h) = gh$$

is a bijection from A to B.

Proposition

If G is a group and $H \subseteq G$ is a subgroup, then the cosets aH and bH are equal ($aH = bH$) if and only if $b^{-1}a \in H$.

Proof. ($\Rightarrow$) Suppose $aH = bH$. Then since $a \in aH$, we have $a \in bH$, so $a = bh$ for some $h \in H$. Hence $b^{-1}a = h \in H$.

($\Leftarrow$) Suppose $b^{-1}a \in H$. Then $a = bh$ for some $h \in H$, so

$$aH = (bh)H = b(hH) = bH.$$

□

Normal Subgroups

A subgroup $N \trianglelefteq G$ is called **normal** if

$$gNg^{-1} = N \quad \text{for all } g \in G.$$

Theorem

The following are equivalent for all $g \in G$:

1. $gNg^{-1} = N$ for all $g \in G$, (aka $N \trianglelefteq G$).
2. $gNg^{-1} \subseteq N$.
3. For all $g \in G$ and $n \in N$, $gng^{-1} \in N$.
4. The left and right cosets of gN and Ng are equal.

Proof. (1) $\Rightarrow$ (2) Immediate.

(2) $\Rightarrow$ (3) Just notation.

(3) $\Rightarrow$ (4) Suppose $gng^{-1} \in N$. We want to prove that $gN = Ng$. Pick any element $gn \in gN$. Since $gng^{-1} \in N$, then $gng^{-1} = n' \in N$, then $gn = n'g$. So $gn = n'g \in Ng$. So $gN \subseteq Ng$. By the same argument, $Ng \subseteq gN$. So $gN = Ng$.

(4) $\Rightarrow$ (1) Assume that $gN = Ng$. Multiply all the elements in the sets on both sides by g^{-1}. $[gN]g^{-1} = [Ng]g^{-1}$. So $gNg^{-1} = N$. $\square$

Quotient Groups

If $N \trianglelefteq G$, we can define the **quotient group** G/N whose elements are the cosets of N in G.

Definition

Define composition in this group by

$$[aN][bN] = (ab)N.$$

We must check that this operation is well-defined.

Well-definedness

Suppose $aN = cN$ and $bN = dN$. Is $(ab)N = (cd)N$? We want: $[aN][bN] = [cN][dN]$. Note: $(ab)N = (cd)N$ if $(cd)^{-1}(ab) \in N$. $(cd)^{-1}(ab) = d^{-1}c^{-1}ab$. Since $aN = cN$, $c^{-1}a \in N$. Let $c^{-1}a = n = d^{-1}nb$. Since N is normal, $bn_2 = n_3b$ for some $n_3 \in N$. Since $bN = dN$, $d^{-1}b \in N$. Since N is normal, $bN = Nb$. So $nb = bn'$ for some $n' \in N$. Hence composition on this group is well-defined.

Example

Let $G = D_4 = \{e, s, s^2, s^3, r, rs, rs^2, rs^3\}$, where $rs = s^{-1}r = s^3r$.

Let $N = \{e, s^2\}$, which is a normal subgroup. N is a group: $(s^2)^2 = s^4 = e$. Check if N is normal. See if $gNg^{-1} \subseteq N$ for any $g \in G$. $e \in N, geg^{-1} = gg^{-1} = e$. How about gs^2g^{-1}? If $g \in D_4$ then $g = r^is^j$. $i = \{0,1\}, j = \{0,1,2,3\}$. $(r^is^j)s(r^is^j)^{-1} = (r^is^j)s^2((s^j)^{-1}(r^i)^{-1}) = (r^is^j)s^2(s^{4-j}r^i) = r^is^2r^i$. Either $i = 0 = s^2$ or $i = 1$ in which case $rs^2r = rssr = s^3rsr = s^6r^2 = s^2$. So $N = \{e, s^2\} \trianglelefteq D_4$.

We check normality:

$$sr^2s = (srs)^2 = r^{-2} = r^2,$$

so $N \trianglelefteq D_4$.

The cosets of N in D_4 are:

$$\begin{aligned} N &= \{e, r^2\}, \\ rN &= \{r, r^3\}, \\ sN &= \{s, sr^2\}, \\ rsN &= \{rs, r^3s\}. \end{aligned}$$

Thus,

$$D_4/N = \{N, rN, sN, rsN\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Coset Multiplication Table

$\cdot$	eN	rN	sN	rsN
eN	eN	rN	sN	rsN
rN	rN	eN	rsN	sN
sN	sN	rsN	eN	rN
rsN	rsN	sN	rN	eN

Hence, $D_4/N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.