MATH 369 Fall 2025 - Class Notes

8/29/2025

Scribe: Name

Summary: We spent the majority of the class going over the **homework**. We talked briefly about **simple linear groups**. We introduced the idea of **normal subgroups** and near the end of class we proved one important theorem about them. We also briefly touched on **cosets**.

Notes:

Part 1: *Homework*

The homework was to find all the homomorphisms from $D_4 \to \mathbb{Z}_4$ and to state the kernel of each one.

We first noted that $|D_4| \neq |\mathbb{Z}_4|$ so none of the homomorphism could find would be bijections, and therefore none of them would be isomorphisms.

We reiterated the fact that homomorphism ensures that $\forall\, a \in$ the domain, $ord(f(a))\Big|ord(a)$.

We used this fact to show that while s might be able to map to any congruence class of $\mathbb{Z}_4$, $ord(r) = 2$ so r could not map to either [1] or [3] since they both had order 4 and $4 \nmid 2$.

We used a guess and check method that didn't turn up any problems for the powers of s and the powers of r individually, but since s and r don't commute and all elements of $\mathbb{Z}_4$ are commutative, we found some issues with some of our choices for $f(s)$. Specifically, the contradictions arose when we chose $f(s) = [1]$ or [3]:

$$sr = rs^3 \implies f(sr) = f(rs^3) \implies f(s) = f(s^3) \implies f(s) = f(s)^3 \implies e = [0] = f(s)^2$$

so $ord(f(s))\Big|\ 2$, but [1] and [3] have order 4, so $f(s) = [0]$ or [2]. Since for each of the two options for $f(r)$ we have two options for $f(s)$, that left us with four working homomorphism:

(a) $f(r) = f(s) = [0]$ with $\ker(f) = D_4$

(b) $f(r) = [0], f(s) = [2]$ with $\ker(f) = \{e, r, s^2, rs^2\}$

(c) $f(r) = [2], f(s) = [0]$ with $\ker(f) = \langle s\rangle$

(d) $f(r) = f(s) = [2]$ with $\ker(f) = \{e, rs, s^2, rs^3\}$

We observed that $\mathbb{Z}_4 \cong \ker(f_d) \ncong \ker(f_b) \cong V_4$, but in all cases $D_4/f \cong \mathbb{Z}_2$

Part 2: *Simple Linear Groups*

We reviewed the idea that $Det : GL_n(\mathbb{R}) \to (\mathbb{R}^\times, \cdot)$ where $Det(M) = |M|$ is a homomorphism.

Since 1 is the identity of the codomain, the kernel of Det is the set of all $n \times n$ matrices with a determinant of 1. We called this kernel the special linear group and denoted it $SL_n(\mathbb{R})$. We noted that since all matricies in this group have a determinant of 1, any product of matrices in this group will also have a determinant of 1, and therefore will preserve area and orientation.

We also saw that $GL_n(\mathbb{R})/Det \cong \mathbb{R}^\times$ since each non-zero real number determinant defines it's own congruence class of matrices.

Part 3: *Normal Subgroups*

We started with a reminder that if H is a subgroup of G, then gHg^{-1} is a subgroup of G for all $g \in G$.

We then used this idea to define a normal subgroup as a special case of the above where $gHg^{-1} = H \quad \forall\, g \in G$. We write $H \trianglelefteq G$ (or $H \triangleleft G$ when we want to also specify $H \neq G$). Think of these symbols a little bit like '$\leq$' and '$<$'.

Theorem 1. *If $f : G \to H$ is a homomorphism, ker$(f) \trianglelefteq G$*

Proof. let $K = \ker(f)$. We want to show that $\forall\, g \in G \quad gKg^{-1} = K$. We will do so by showing mutual inclusion.

Show: $gKg^{-1} \subseteq K$
Let $x \in gKg^{-1}$
$f(x) = f(g)ef(g^{-1}) = f(gg^{-1}) = e$
Since f maps x to the identity, $x \in K$
$\therefore\ gKg^{-1} \subseteq K$

Show: $K \subseteq gKg^{-1}$
Let $y \in K$. Since $g^{-1} \in G$, we just showed that $g^{-1}yg \in K$.
$\therefore\ g(g^{-1}yg)g^{-1} \in gKg^{-1}$.
Since $g(g^{-1}yg)g^{-1} = (gg^{-1})y(gg^{-1}) = eye = y$, $y \in gKg^{-1}$.
$\therefore\ K \subseteq gKg^{-1}$

$\therefore\ gKg^{-1} = K$, which means that $\ker(f) \trianglelefteq G$

$\square$

Part 4: *Cosets*

If H is a subgroup of G and $a \in G$, we called aH a *left coset* of H.

We noted that cosets of H are sub*sets* of G but are usually not sub*groups* of G since only the trivial coset $hH = H$ will contain the identity.

We explored the example $G = \mathbb{Z}_{12}$, $H = \langle 3 \rangle$ and saw that the left cosets of H were:

(a): $0 + \langle 3 \rangle = \{0+0, 0+3, 0+6, 0+9\}$

(b): $1 + \langle 3 \rangle = \{1+0, 1+3, 1+6, 1+9\}$

(c): $2 + \langle 3 \rangle = \{2+0, 2+3, 2+6, 2+9\}$

Since G is abelian, the right cosets are the same as the left cosets.

We then explored another example where G was not abelian and found that the right and left cosets don't have to match when G is not abelian.

Consider $H = \langle (1,2) \rangle$, $G = S_3$.

Left cosets:

(L_1): let $a_1 = ()$, $()H = \{(), (1,2)\}$

(L_2): $(2,3) \notin a_1 H$, so let $a_2 = (2,3)$, $(2,3)H = \{(2,3), (1,3,2)\}$

(L_3): $(1,3) \notin \bigcup_{i=1}^{2} a_i H$, so let $a_3 = (1,3)$, $(1,3)H = \{(1,3), (1,2,3)\}$

Right cosets:

(R_1): let $a_1 = ()$, $H() = \{(), (1,2)\}$

(R_2): $(2,3) \notin Ha_1$, so let $a_2 = (2,3)$, $H(2,3) = \{(2,3), (1,2,3)\}$

(R_3): $(1,3) \notin \bigcup_{i=1}^{2} Ha_i$, so let $a_3 = (1,3)$, $H(1,3) = \{(1,3), (1,3,2)\}$

There are the same number of left and right cosets, and all the cosets have the same size, and $L_1 = R_1 = H$. However, in this example, when $i \neq 1$, $L_i \neq R_j$ for any j. Similarly, $j \neq 1 \implies R_j \neq L_i \;\; \forall i$.