MATH 369 Fall 2025 - Class Notes

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Summary: This class was mostly about homomorphisms. We also touched a little bit on normal subgroups, and left and right cosets.

Homomorphisms

We consider homomorphisms from D_4 to $\mathbb{Z}_4$.

Basic Definitions

- $|D_4| = 8$
- $|\mathbb{Z}_4| = 4$

A **homomorphism** $f : G \to H$ satisfies

$$f(xy) = f(x)f(y) \quad \text{for all } x, y \in G.$$

Example: $D_4 = \langle r, s \mid r^4 = s^2 = e,\, srs = r^{-1} \rangle$.

Identity and Kernel

The function that sends everything to the identity is a homomorphism.

The **kernel** of a homomorphism f is

$$\ker(f) = \{g \in G \mid f(g) = e_H\}.$$

This is always a normal subgroup of G.

If $\ker(f) = \{e\}$, then f is injective.

Example: $f : D_4 \to \mathbb{Z}_4$

We must have $f(r)^4 = 0 \mod 4$ and $f(s)^2 = 0 \mod 4$.

We try the following:

$$f(r) = 1, \quad f(s) = a.$$

Then,

$$f(srs) = f(r^{-1}) \implies f(s) + f(r) + f(s) = -f(r) \pmod 4.$$

That is,

$$a+1+a \equiv -1 \pmod 4 \Rightarrow 2a+2 \equiv 0 \pmod 4.$$

Hence $a \equiv 1$ or $3 \pmod 2$. Only $a \equiv 0$ or 2 works. Testing $a = 0, 2$:

$$f(s) = 0,\ f(r) = 1 \quad \Rightarrow \text{homomorphism holds.}$$

So f is valid if $f(r) = 1$ and $f(s) = 0$.

$$\ker(f) = \{e, r^2, s, sr^2\}.$$

Thus,

$$D_4/\ker(f) \cong \mathbb{Z}_2.$$

Example: $f : D_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$

We can send:

$$f(r) = (1, 0), \quad f(s) = (0, 1).$$

Then:

$$f(r^2) = (0, 0), \quad f(srs) = f(r^{-1}) = f(r).$$

The homomorphism holds.

$$\ker(f) = \{e, r^2\}, \quad D_4/\ker(f) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Linear Algebra Groups

Determinant Map

$$\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times, \quad A \mapsto \det(A).$$

$$\det(AB) = \det(A)\det(B).$$

The kernel is:

$$\ker(\det) = \{A \in GL_n(\mathbb{R}) \mid \det(A) = 1\} = SL_n(\mathbb{R}),$$

the **special linear group**.

Hence,

$$GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R}^\times.$$

Normal Subgroups and Homomorphisms

Theorem 1. *If $f : G \to H$ is a homomorphism, then* $\ker(f) \trianglelefteq G$.

Proof. Let $x \in \ker(f)$ and $g \in G$. Then

$$f(gxg^{-1}) = f(g)f(x)f(g)^{-1} = f(g)e_H f(g)^{-1} = e_H.$$

So $gxg^{-1} \in \ker(f)$. $\square$

Quotient Groups and Cosets

If H is a subgroup of G, the left cosets of H in G are:

$$aH = \{ah \mid h \in H\}.$$

Similarly, right cosets are:

$$Ha = \{ha \mid h \in H\}.$$

If G is abelian, left and right cosets are the same.

Example: $\mathbb{Z}_6$ and $\mathbb{Z}_3$

Let $G = \mathbb{Z}_6$ and $H = \{0, 3\}$.

Then the left cosets of H are:

$$\begin{aligned} 0 + H &= \{0, 3\}, \\ 1 + H &= \{1, 4\}, \\ 2 + H &= \{2, 5\}. \end{aligned}$$

Thus, $\mathbb{Z}_6/H \cong \mathbb{Z}_3$.

Example in S_3

Let $H = \langle (1\,2) \rangle$. Then the left cosets of H are:

$$\{e, (1\,2)\}, \quad \{(1\,3), (1\,2\,3)\}, \quad \{(2\,3), (1\,3\,2)\}.$$

Right cosets are:

$$\{e, (1\,2)\}, \quad \{(1\,3), (1\,3\,2)\}, \quad \{(2\,3), (1\,2\,3)\}.$$

Since S_3 is non-abelian, left and right cosets differ.