MATH 369 Fall 2025 - Class Notes

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Summary: Today's class began by showing that the converse of Lagrange's Theorem is false using the alternating group of 4 elements A_4. We then discussed homomorphisms and looked at some examples of them.

Notes:

Continued from end of last class.

Claim: A_4 does not have a subgroup of order 6. This claim demonstrates that the converse (converse of "if a then b" is "if b then a) of Lagrange's Theorem is false. This is not always a true statement which is why iff statements need to be proven both ways.

Theorem 1. *Lagrange's Theorem: If G has a subgroup H, then the order of H is a divisor of the order of G.*

Lagrange's theorem is not iff. Alternate statement of Lagranges theorem:"If G has a subgroup H, then $d = |H|$ is an integer which divides the integer $n = |G|$.

Converse of Lagrange's Theorem: If d is a divisor of $n = |G|$, then G has a subgroup H with $|H| = d$.

This converse is false as $|A_4| = 12$, but A_4 has no subgroup with 6 elements.

Last class we saw that A_4 has 3 elements of order 2 (all of the form $(i\ j)(k\ l)$), 8 elements of order 3 (all 3 cycles), plus the identity.

To prove this, we must first prove a proposition: If G has even order, then it has an odd number of elements with order 2.

Proof. Let $S = \{x \in G \mid x \neq x^{-1}\}$, the set of all elements with order bigger than 2.

$|S|$ is even since every element can be paired with its inverse (which is not itself).

$|G \setminus S| = |G| - |S|$ is even (even - even = even). The identity $e \in G \setminus S$, so the number of elements with order 2 is $|G \setminus S| - 1$, an odd number. □

Corollary: Every group with even order has at least one element with order 2.

Back to the claim that A_4 does not have a subgroup with 6 elements.

Proof. Suppose $H \subseteq A_4$ with 6 elements, $|H| = 6$. By previous proposition, H has an odd number of elements with order 2. So, it has either 1 or 3 elements with order 2, since A_4 only has 3 elements with order 2.

Case 1: Let's suppose it has 3 elements of order 2 (must have all 3 elements of the form $(i\ j)(k\ l)$) in A_4). The remaining two elements are a 3-cycle, say $(x\ y\ z)$ and $(x\ y)(z\ w)$. So, $(x\ y)(z\ w) \cdot (x\ y\ z) \in H$

$(x\ y)(z\ w) \cdot (x\ y\ z) = (x)(y\ w\ z) = (y\ w\ z)$ which is not in H; a contradiction.

Case 2: Suppose that H has exactly one element of order 2, call it a. So, everything else has order 3. Let b be one element of order 3 in H. Then $b^{-1} \in H$ has order 3.

Let c be another element of order 3, $c \neq b$ and $c \neq b^{-1}$. Then $c^{-1} \in H$ has order 3.

$H = \{e, a, b, b^{-1}, c, c^{-1}\}$

What is bc? It has to be something in the group.

$bc \neq e$; since $c \neq b^{-1}$

$bc \neq b$; since $c \neq e$ (left cancellation)

$bc \neq c$; since $b \neq e$ (right cancellation)

$bc \neq b^{-1}$; since $b^3 = e$ (b has order $3, b^2 = b^{-1},$ but $c \neq b$, so this does not hold.

$bc \neq c^{-1}$; since $c^{-1} = c^2$, right cancel the c's you get $b = c$, but $c \neq b$.

We have ruled out everything except for a. Som $bc = a$. By the exact same argument, $bc^{-1} = a$. So, $bc = a = bc^{-1}$, left canceling the b's we get: $c = c^{-1}$, which is a contradiction since c has order 3, but this implies c has order 2.

Since H cannot have 1 or 3 elements of order 2, there cannot be a subgroup of order 6. $\square$

Homomorphisms:

Isomorphisms without the bijection requirement.

Homomorphisms give us a whole new collection of tools we can use to study groups.

Ex: Not one-to-one, but onto

$Det : GLn(\mathbb{R}) \to \mathbb{R}^{\times}$

$Det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 = Det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right).$

Since $Det(AB) = Det(A)Det(B)$, Det is a homomorphism.

$F(ab) = F(a)F(b) \quad \forall a, b \in G$ is called the homomorphism property.

Ex: $\phi\mathbb{Z} \to \mathbb{Z}_n$

$\phi(a) = [a]_n$

has the homomorphism property. Not isomorphic since it is not one-to-one, but onto.

Ex: one-to-one, but not onto $f : \mathbb{R}^{\times} \to GL_n(\mathbb{R})$

$f(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

One-to-one, but not onto $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ since is not in image.

$f(ab) = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = f(a)f(b)$

So it has the homomorphism property.