MATH315-Notes 09/27/17

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1 Introduction

2 Notes 27 September 2017

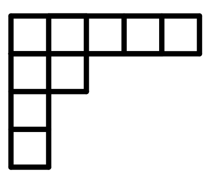
(Review definition of partitions from last class)

2.1 Ferrers Board

Ferrers Board - a useful way to visualize a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$.

The Ferrers board of λ has λ_1 boxes in the first row, and λ_2 boxes immediately below (in the second row), and so on and so forth.

Ex.: A partition of 9 would be λ = (5,2,1,1)



Visualize it as how many ways you can arrange/place the blocks around., so that the sum of the blocks is the original number.

2.2 Conjugates and Self-Conjugates

Definition: **The conjugate** λ^c of a partition λ is the partition that comes from flipping the corresponding Ferrers Board across the main diagonal (columns turn into rows).

2.2.1 Theorem:

The number of **partitions** of n into exactly k parts $\big(p_k(n)\big)$ is equal to the number of partitions of n whose largest part is equal to k.

Reminder: bijection is one to one and onto. (prove by finding inverse)

Proof:

Take a partition of n with k parts, and take its conjugate. The conjugate has largest part k.

The conjugate operation is a bijection, since conjugating twice gives the original partition.

So every partition with exactly k parts corresponds to a partition whose largest part has size k.

Definition: A partition is called self-conjugate if it is equal to its own conjugate (symmetric across its main diagonal).

2.2.2 Theorem:

The number of **self-conjugate** partitions of n is equal to the number of partitions of n into distinct odd parts.

(We can only write down odd numbers, and can't use any twice)

Try this for n=5:

Odd distinct parts: 5

Try for n=7:

Odd distinct parts: 7

Try for n=8:

Odd distinct parts: 5+3, 7+1

Self-conjugate ways to do it: (4, 2, 1, 1), (3, 3, 2)

Proof:

Write down a **bijection** from a **self-conjugate** partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ to a partition with distinct odd parts.

Remove "hooks" (entire first row and first column from the self-conjugate partition) one at a time, turning each one into a row of the new partition. (Take our entire row and column of partition, and peel it off.)

What's left over will still be a **self-conjugate** partition, so we can repeat this process over and over again, until all the elements in the Ferrers board are removed.

The resulting partition will be $((2\lambda_1 - 1), (2\lambda_2 - 3), (2\lambda_3 - 5)$.

Note: (The top square was both a row and a column, so we counted twice.)

Lose the (2,1) entry and (3,1) entry because they were from the first hook, and lose "corner box" from double-counting.

All these numbers are ODD. We want to argue that they are all distinct. Because each successive λ_i is bigger than the last, we subtract larger numbers each time.

The entries are distinct because the λ_i are non-increasing.

We need to now show that it is a **bijection.** Note that if we have a partition into distinct odd parts.

Say we have, $\sigma = (\sigma_1, \sigma_2, ..., \sigma_k)$. Since the σ_i are odd, we can write $\sigma_i = 2s_i - (2i - 1)$ for odd j.

Take the ith element of σ, and subtract the ith off number. Divide by 2 to get s_i.

This gives us a sequence of numbers s_i, which can be used to form the rows of a self-conjugate partition, which will map to σ under our bijection.

So, our bijection is onto. If two self-conjugate partitions map to the same partition into odd distinct parts, then all their "hooks" had the same size.

Every single one of the "hooks" that showed up in our self-conjugate partitions was equal.

But, the hooks uniquely describe a self-conjugate partition (same number of elements in rows as columns).

2.3 More Partitions!!

2.3.1 Theorem:

Let $q(n) =$ the number of partitions of n into parts of size at least 2.

$$q(n) = p(n) - p(n - 1)$$

Proof:

Note that any partition of n that contains a part of size 1 corresponds to a unique partition of $(n-1)$ obtained by removing the (last) piece of size 1.

Thus, the total number of partitions of n that contain at least one 1 is counted by $p(n-1)$. So, if we take $p(n) - p(n-1)$, this counts all partitions of n removing the ones which contain at least one 1, leaving only those with parts all at least 2.

In effect, we took the entire set, and subtracted off the complement.

3 Not-So Vicious Cycles in Permutations!

(what follows is not covered on the midterm on 10/04)

3.1 Permutations and Counting

<u>How many ways are there to write the numbers 1 through n around a circle?</u> The only thing that matters is what numbers are next to a given number.

(seats are not distinguished; rotate the table, nothing changes)

3.1.1 Theoremm:

This is a cycle of the numbers 1 through n. The number of cycles of $[n]$ is $(n-1)!$.

Proof:

There are $n!$ ways to write the numbers into the n positions around a fixed circle. (say we can tell whose seats are whose; there are n ways to rotate the circle)

Now there are n different ways to rotate the circle, and get the same cycle.

So, every cycle was counted n different times. Total of cycles is $\frac{n!}{n} = (n-1)!$.

Permutation is a rule that maps $[n] \mapsto [n]$: one-to-one and onto.

We can apply the permutation multiple times. Permutation π: π^2 means take $\pi^2(i) = \pi(\pi(i))$. In general, π^k is the permutations we get if we apply π to itself k times.