MATH315-Notes 09/25/17

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1 Stirling Numbers of the Second Kind

$S(n, k)$ = the number of ways to partition $[n]$ into k nonempty subsets

From last class, we found that S(n,k) = S(n-1, k-1) + k(n-1, k).

If $k > n$, you have more boxes than you have things, so this is not possible. Then, $S(n, k) = 0$.

Also, $S(n, 0) = 0$ (putting a positive n objects into 0 boxes).

1.1 Compute Small Values of S(n,k)

$\frac{k}{n}$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

Notes: $S(2, 1) = S(1, 0) + 1 * S(1, 1)$

$S(3, 2) = S(2, 1) + 2 * S(2, 2)$

1.2 Definition of the n-th Bell Number:

For a fixed n, $\mathbf{B(n)} = \sum_{k=1}^{n} S(n, k)$ is the total number of ways to partition $[n]$ into any number of parts. This is called the n-th Bell number.

The number of ways to separate 1 object is one, thus $B(1) = 1$.

B(n)	Value of B(n)
$B(2)$	$1+1=2$
$B(3)$	$1+3+1=5$
$B(4)$	$1+7+6+1=15$
$B(5)$	$1+15+25+10+1=52$
$\vdots$	$\vdots$
$B(n)$	$\sum_{k=1}^{n} S(n,k)$

1.3 Surjective Functions

Theorem:

The number of surjective (onto) functions from $[n] \to [k]$ is $k! * S(n,k)$,

Proof:

Any surjective function f : $[n] \to [k]$ gives a partition of n into exactly k boxes, by assigning i to be placed in box $f(i)$.

None of the k boxes is empty, since f is surjective.

Now, we have to assign labels to previously unlabeled boxes (from 1 to k), and the number of ways to do so is $k!$.

Since the k boxes can be assigned labels from 1 to k in $k!$ different ways, any set partition of $[n]$ into exactly k nonempty boxes corresponds to $k!$ different surjective functions.

Note that if f is a bijection. then $n = k$ and $S(n,k) = 1$.

So, our formula gives $k! * S(n,k) = k!$ which is the number of permutations.

Recall the number of ways to place n distinct objects into k distinct boxes (possibly empty) was k^n.

How could we rewrite the above in terms of **functions?** This counts the **number of functions** f from $[n] \to [k]$.

Theorem:

$$k^n = \sum_{i=1}^{n} S(n,i) \frac{k!}{(k-n)!}$$

Proof:

How would we prove this?

The **left-hand side** is counting the number of ways to put n distinct objects into k distinct functions. ($f : [n] \to [k]$)

Not quite a Bell number, but close. If $i = 1$, then we have one box.

The **right-hand side** counts functions $f : [n] \to [k]$ based on how many elements are in the image, or based on how many boxes have something in them.

Fix i, where $1 \leq i \leq n$. *We need to use the Sterling Numbers, in some way.*

We want to count surjective functions from $[n]$ to a subset of $[k]$ of size i. There are $\binom{\mathbf{k}}{\mathbf{i}} = \frac{\mathbf{k!}}{\mathbf{i!(k-1)!}}$ subsets of **k** of size **i**. Additionally, there are **S(n,i)*i!** surjective functions from **[n]** to a set of size **i**.

So, we get..

$$S(n,i)\frac{k!}{(k-1)!}$$

..possible functions onto a set of size i.

Now, sum over i.

If both things are identical, . . .

If both the objects and the boxes are **identical**, we get integer partitions.

The only important thing is the number of objects that are in a box together.

Since the order of the boxes doesn't matter, we can just record the number of objects in each box, writing the numbers in decreasing order.

Example: find all integer partitions of 4. (4 things all together), (2,2), (3,1), (2,1,1), (1,1,1,1)

A **partition** of n is any sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_k$, and $\lambda_1 + \lambda_2 + ... + \lambda_k = n$.

If k is fixed, we call this a partition of size k.

Notation:

$p(n)$ = the number of integer partitions of n

$p_k(n)$ = the number of integer partitions of n into exactly k parts

No one knows a formula to compute $p(n)$ for any value of n, but we do have the ability to sum $p_k(n)$.

$$p(n) = \sum_{k=1}^{n} p_k(n)$$

2 Theorem (Hardy, Ramanujan)

(see *The Man Who Knew Infinity*!)

2.1 Asymptotic Expression of p(n)

$$P(n) \sim \frac{1}{4\sqrt{3}} e^{\pi\sqrt{\frac{2\pi}{3}}}, n \to \infty$$

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