MATH 315 - Class Notes

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Summary:

- Multinomial coefficients
- The General Format of Combinatorial Placement based on distinct vs. identical objects and boxes
- Set Partitions

Multinomial Coefficients:

$$(x_1 + x_2 + ...x_k)^n = \sum_{m_1+m_2+...m_k=n} \binom{n}{m_1 + m_2 + ...m_k} * x_1^{m_1} * x_2^{m_2} * ...x_k^{m_k}$$

Below represents the number of ways to pick n things where $m_1$are the first object, m_2 of second...m_k of the last

$$\binom{n}{m_1 + m_2 + ...m_k}$$

$$\binom{n}{m_1 + m_2 + ...m_k} = \frac{n!}{(m_1!)(m_2!)...(m_k!)}$$

General Format of Combinatorial Placement:

- n boxes
- k objects to place in these boxes

Consider if the objects or boxes distinct/not distinct

		Boxes	
		Distinct	Identical
Objects	Distinct	125	5
	Identical	35	3

Try to work out math for the chart when n=5,k=3
The bullets below are represented as Objects x Boxes:

- **Distinct x Distinct**

$$\text{Each object has 5 boxes it can be placed into}$$
$$\text{5 options for object A}$$
$$\text{5 objects for object B}$$
$$\text{5 objects fro object C}$$
$$\text{Therefore:}$$
$$5 * 5 * 5 = 5^3 = 125$$
$$n^k = 5^3 = 125$$

- **Distinct x Identical**

  There are five combinations total: (A B C) (AB C) (AC B) (BC A) (ABC)

- **Identical x Distinct**

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Stars and Stripes Problem:
How many ways can you write down a sequence containing k bars and n-k stars?

/**/****/**
* = objects
/ = we need one fewer bars than boxes

Based on the notation in the book, let n = objects and k = boxes
Using the stars and bars example, the number of ways to put n identical objects into k distinct boxes is:

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Any way to do this is called a weak composition of n into k parts
A <u>Composition</u> is a weak composition where every part has size at lease one with no empty boxes
A <u>Weak Composition</u> can have empty boxes, but compositions can't have empty boxes

How can we count compositions?
Put one object in each box first. This leaves us with:

$$\binom{n-1}{k-1}$$

You get this binomial because there are k boxes.

How many compositions of n are there in total (of any size)?
There has to be at least one box and at most there are n boxes if every object gets its own box.

Summing over all possible values of k we get:

$$\sum_{k=0}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$$

This works because of the Binomial Theorem:

$$(x+y)^n = \sum_{i=1}^{n} \binom{n}{i} * x^i * y^{n-i}$$

Set x=y=1 in order to get rid of:

$$x^i * y^{n-i}$$

This then gives you:

$$= \sum_{i=0}^{n} \binom{n}{i} = 2^n$$

This allows you to sum an entire row of Pascal's triangle

Proof 2:
There are 2^{n-1} total compositions of n.

By induction if n=1 (Base Case):
$2^{n-1} = 2^{1-1} = 2^0 = 1$ because there is only one way to put an object in a box.

Induction Hypothesis:
Assume there are 2^{n-1} total compositions of n.

Induction Step:
We want to prove this for n+1.
Take n of the objects and make a composition of them.
By the Induction Hypothesis, there are 2^{n-1} ways to do this.
For each composition we will either:

1. Put one more object into the first box allowing box 1 to have at least 2 objects in it
2. Create a new box at the beginning and put the last object into it.

Note that every composition of n+1 is created from a composition of size n in exactly one way.
So, every composition of n corresponds to 2 compositions of n+1
This gives you: $2^n = 2^{(n+1)-1}$ compositions total of size n+1

- **Identical x Identical**

There are only three different subsets.
Either all objects can go in one box, two objects can go in one box and one object in another box, or all objects in a seperate box
Possible options: 3, 2+1, 1+1+1

Set Partitions:

Set partitions are ways to distribute distinct objects into identical boxes

Take k $\leq$ n
Write S(n,k) to be the number of ways to put n distinct objects into k nonempty identical boxes.

S(n,k)
S(3,1)=1
S(3,2)=3
S(3,3)=1

These numbers are called stirling numbers of the second kind.

What if n=4, k=2?

Compute S(4,2)=7
Possible Sets:
a bcd
b acd
c abd
ab cd
ac bd
ad bc

Theorem:
S(n,k)= S(n-1,k-1)+ kS(n-1,k)

Proof:
LHS: counts ways to put the numbers 1 through n in k non empty identical boxes
RHS: counts the same thing but pay attention to the largest element n. (Everything else forms a set partition of the numbers 1 through n-1)

If n is in a box by itself, then n-1 is partitioned into k-1 boxes S(n-1,k-1)

Otherwise, we are left with a partition of n-1 into k boxes S(n-1,k)
There are k options for where the object n can go so we get kS(n,k)

Adding them together gives: S(n-1,k-1)+kS(n-1,k)

Note that:
S(n,k)=0 if k¿n
This lets us compute S(n,k) using a table like pascal's triangle.

k

S(n,k)	1	2	
1	1		
2	1	1	
3	1	3	1