MATH 315 - Class Notes

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Summary: Combinatorial Structures including permutations, combinations, and pascal's triangle.

Permutation: A way to organize the numbers **1...n** in different ways. **n!** ways to do this.
n distinct objects distributed into **n** distinct boxes so that each box has **1** object. Think of a permutation as a bijection from **[n]-¿[n]** where **[n]=1,2,3,...,3** (Bijection meaning one-to-one and onto).

Combination: Pick an ordered set of **k** objects from **n** objects.
Notation:

$$n_k = n(n-1)(n-2)...(n-k+1)$$

k distinct boxes, each containing **1** of a set of **n** distinct objects (care about the order).
Often we don't care about the order. Boxes are no longer distinct so divide the equation by the number of ways to rearrange the objects.
Formula:

$$\frac{nk}{k!} = \frac{n(n-1)(n-2)...(n-k+1)}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

This is the binomial coefficient.
Pascal's Triangle: Put all binomial coefficients where there are **m** objects total in row **m** having increasing values of **k**.

$n = 0$:					1				
$n = 1$:				1		1			
$n = 2$:			1		2		1		
$n = 3$:		1		3		3		1	
$n = 4$:	1		4		6		4		1

Patterns in Pascal's Triangle:

- Rows are palindromes. (Triangle is symmetric).

$$\binom{n}{k} = \binom{n}{n-k}$$

- A binomial coefficient is the sum of the 2 entries above it in Pascal's triangle.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Combinatorial Proof:
RHS: number of ways to choose k things from [n].
LHS: counts the same thing, but conditioning on whether n is included in the set or not.

Subsets of [n] of size k which include n: $binomn-1k-1$ways to do this. Subset of [n] of size k which do not include k: $binomn-1k$ways to do this. These are disjoint and every subset of [n] size k appears exactly once. Q.E.D.

Hockey Stick Pattern:

$$\binom{n}{0}+\binom{n+1}{1}+...+\binom{n+k}{k}=\binom{n+k+1}{k}=\sum_{i=0}^{k}\binom{n+1}{i}$$

Combinatorial Proof:
RHS: $\binom{n+k+1}{k}=\binom{n+k+1}{n+1}$ counts the ways to choose n+1 things from n+k+1 things.
LHS: count the same thing (subsets of [n+k+1] of the size n+1) conditioned on the largest element.
How many ways are there to pick n+1 things from [n+k+1] where the largest object is exactly n+1?
$\binom{n+i-1}{n}$ the total number of ways to do this:

$$\sum_{i=1}^{k+1}\binom{n+i-1}{n}=\sum_{i=1}^{k+1}\binom{n+i-1}{i-1}=\sum_{i=0}^{k}\binom{n+i}{i}$$

Q.E.D.

$$\binom{n}{0}^2+\binom{n}{1}^2+...+\binom{n}{n}^2=\binom{2n}{n}$$

Combinatorial Proof:
RHS: Choose n things from [2n].
LHS: Choose n things from [2n] conditioned on the number of things coming from 1,2,...,n.
Suppose i things came from [n]. $0 \leq i \leq n$. $\binom{n}{i}$ ways to choose i things from [n]. We still have to choose n-i things from n+1,...,2n. $\binom{n}{n-i}=\binom{n}{i}$ ways to do this. These two choices are independent so $\binom{n}{i}^2$ ways to do this. These are distinct and count every subset of [2n] of size n.

$$\sum_{i=0}^{n}\binom{n}{i}^2$$

Q.E.D.
Binomial Theorem:

$$(x+y)^n=\sum_{i=0}^{n}\binom{n}{i}x^iy^(n-1)$$

Proof: Foil out $(x+y)^n=(x+y)(x+y)...(x+y)$. Each multiplication whithout letting x and y commute gives a sequence of n letters that are either x or y. Now group the terms together with the same number of x's. How many terms have i x's? $\binom{n}{i}$ ways. So each term $x^iy^(n-1)$ appears $\binom{n}{i}$ many times. Q.E.D.