Class Notes for 10-25-17

Suppose that we have two generating functions A(x) and B(x), where:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n$$

Suppose that we define a function C(x), such that C(x) = A(x) + B(x), then:

$$C(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n$$

$$C(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

What happens when we we to multiply A(x) and B(x)?
Let us define: C(x) = A(x)*B(x) where $C_n = \sum_{n=0}^{\infty} c_n x^n$ Clearly, we can tell that $c_n \neq a_n b_n$

Let us try and determine an expression for $c_n$:

$$C(x) = (a_0 + a_1 x^1 + a_2 x^2 + ....)(b_0 + b_1 x^1 + b_2 x^2 + ...)$$

$$C(x) = (a_0 b_0) x^0 + (a_0 b_1 + a_1 b_0) x^1 + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + ...$$

From this, we can see that the general expression for $c_n$ is:

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

So, for C(x) = A(x)*B(x), we get that:

$$C(x) = \sum_{n=0}^{\infty} (\sum_{i=0}^{n} a_i b_{n-i}) x^n$$

Example: We know that: $\frac{1}{1-x} = \sum_{n=0}^{\infty} 1x^n$ Suppose that $A(x) = \frac{1}{1-x}$, and we want to compute $A(x) \times A(x)$.

$$A(x) * A(x) = (\frac{1}{1-x})(\frac{1}{1-x}) = \frac{1}{(1-x)^2}$$

Now, find the power serires for $A(x)^2$:

$$A(x)^2 = \sum_{n=0}^{\infty}(\sum_{i=0}^{n} a_i a_{n-i})x^n$$

$$A(x)^2 = \sum_{n=0}^{\infty}(\sum_{i=0}^{n}(1))x^n$$

$$A(x)^2 = \sum_{n=0}^{\infty}(n+1)x^n$$

Example: What is the generating function for $\sum_{n=0}^{\infty} nx^n$?

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$$

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^{n+1}$$

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty}(n)x^n$$

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty}(n)x^n + (0)(x^0) - (0)(x^0)$$

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty}(n)x^n$$

Thus, the generating function for $\sum_{n=0}^{\infty} nx^n$ is:

$$\frac{x}{(1-x)^2}$$

Example: Suppose that we want to find a generating function for: $\sum_{n=0}^{\infty}(n+1)x^n$.
We know that:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n ========> (1)$$

Now, we will take the derivative of both sides of(1):

$$\frac{d}{dx}(\frac{1}{1-x}) = \frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\frac{d}{dx}(\sum_{n=0}^{\infty} x^n) = \frac{d}{dx}(x^0) + \frac{d}{dx}(\sum_{n=1}^{\infty} x^n)$$

$$=> 0+\sum_{n=1}^{\infty} nx^{n-1}$$

$$=> \sum_{n=0}^{\infty}(n+1)x^n$$

Thus, our generating function for $\sum_{n=0}^{\infty}(n+1)x^n$ is $\frac{1}{(1-x)^2}$

Back to the Binomial Theorem:

$$(x+1)^m = \sum_{n=0}^{m} \binom{m}{n} x^n$$

Think of this as a generating function for the binomial coefficients: $a_n = \binom{m}{n}$, where this is 0 whenever n is greater than m.

The Generalized Binomial Theorem (for any real number $\alpha$):

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

Define $\alpha$ as follows:

$$\binom{\alpha}{n} = \frac{(\alpha)(\alpha-1)(\alpha-2)...(\alpha-n+1)}{n!}$$

Note: This definition gives the same expression as usual if$\alpha$ is an integer.

Example: $(\alpha=-2)$:

$$\frac{1}{(1-x)^2} = (1-x)^{-2}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \binom{-2}{n} (-x)^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \frac{(-2)(-3)(-4)...(-2-n+1)}{n!} (-x)^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n(2)(3)(4)...(n+1)}{n!} (-1)^n(x^n)$$

We know that $(-1)^n*(-1)^n = (-1)^{2n} = 1$ and there is a n! in the numerator and denominator. So we are left with:

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$$

Example: Find a closed form for $a_n = 6a_{n-1} - 9a_{n-2}$ where $a_0 = 1$ and $a_1 = 9$. Let us define $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Our first step is to multiply both sides by $x^n$.

$$a_n x^n = (6a_{n-1} - 9a_{n-2})x^n$$

Our next step is to sum over all values of $n$.

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} 6a_{n-1}x^n + \sum_{n=2}^{\infty} 9a_{n-2}x^n$$

Our next step is so modify the sums so that they match up with our definition of $A(x)$. Examining each term separately, we get:

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_n x^n + a_0x^0 + a_1x^1 - a_0x^0 - a_1x^1$$

$$\sum_{n=2}^{\infty} a_n x^n = A(x) - 1 - 9x$$

$$\sum_{n=2}^{\infty} 6a_{n-1}x^n = 6\sum_{n=2}^{\infty} a_{n-1}x^n$$

$$\sum_{n=2}^{\infty} 6a_{n-1}x^n = 6x\sum_{n=2}^{\infty} a_{n-1}x^{n-1}$$

$$\sum_{n=2}^{\infty} 6a_{n-1}x^n = 6x\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$$

$$\sum_{n=2}^{\infty} 6a_{n-1}x^n = 6x\sum_{n=1}^{\infty} a_n x^n + a_0x^0 - a_0x^0$$

$$\sum_{n=2}^{\infty} 6a_{n-1}x^n = 6x(A(x) - 1)$$

$$\sum_{n=2}^{\infty} 9a_{n-2}x^n = 9\sum_{n=2}^{\infty} a_{n-2}x^n$$

$$\sum_{n=2}^{\infty} 9a_{n-2}x^n = 9x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2}$$

$$\sum_{n=2}^{\infty} 9a_{n-2}x^n = 9x^2\sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} 9a_{n-2}x^n = 9x^2 A(x)$$

Now, plugging all of these back in to our original expression, we have:

$$A(x) - 1 - 9x = (6A(x) - 6x) - 9x^2A(x)$$
$$A(x) - 6xA(x) + 9x^2A(x) = 1 + 9x + -6x$$
$$A(x)(1 - 6x + 9x^2) = 1 + 3x$$
$$A(x) = \frac{1+3x}{1-6x+9x^2} = \frac{1}{(1-3x)^2} + \frac{3x}{(1-3x)^2}$$

Our final step is to solve for a closed form of $a_n$.

$$A(x) = \frac{1}{(1-3x)^2} + \frac{3x}{(1-3x)^2}$$
$$A(x) = \sum_{n=0}^{\infty}(n+1)(3x)^n + 3x\sum_{n=0}^{\infty}(n+1)(3x)^n$$
$$A(x) = \sum_{n=0}^{\infty}(n+1)(3^n)(x^n) + \sum_{n=0}^{\infty}(n+1)(3^{n+1})x^{n+1}$$
$$A(x) = \sum_{n=0}^{\infty}(n+1)(3^n)(x^n) + \sum_{n=1}^{\infty}(n)(3^n)x^n$$
$$A(x) = (0+1)3^0x^0 + \sum_{n=1}^{\infty}(n+1)3^nx^n + \sum_{n=0}^{\infty}n3^nx^n$$
$$A(x) = 1 + \sum_{n=1}^{\infty}3^n(2n+1)x^n$$
$$a_n = 3^n(2n+1)$$

Which is our closed form for $a_n$.