Class notes for 10-23-17

Review: Generating Functions Example: Tower of Hanoi.
The number of moves required for n discs is:

$$a_{n+1} \; where \; a_{n+1} = 2a_n + 1$$

Our first step is to multiply both sides of the recurrence by x^n
Which yields:

$$a_{n+1}x^n = 2a_nx^n + x^n$$

Our goal is to find $A(x) = \sum_{n=0}^{\infty} a_nx^n$

Next, we will sum both sides:

$$\sum_{n=0}^{\infty} a_{n+1}x^n = \sum_{n=0}^{\infty} 2a_nx^n + \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{x} \sum_{n=0}^{\infty} a_{n+1}x^{n+1} = 2A(x) + \frac{1}{1-x}$$

In this case, to "match" the left hand side of the equation to A(x), we need to change the indice to n=0. So we will add and subtract an "a_0" term.

$$\frac{1}{x} \sum_{n=1}^{\infty} a_{n+1}x^{n+1} + \frac{a_0}{x} \; - \frac{a_0}{x} \; = 2A(x) + \frac{1}{1-x}$$

$$\frac{A(x)}{x} \; - \frac{0}{x} \; = 2A(x) + \frac{1}{1-x}$$

$$A(x) - 2xA(x) = \frac{1}{1-x}$$

$$A(x)(1-2x) = \frac{1}{1-x}$$

$$A(x) = \frac{x}{(1-x)(1-2x)}$$

We know that:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \; = 1 + x^2 + x^3 + ...$$

Which means:

$$\frac{1}{1-ax} \; = \sum_{n=0}^{\infty} a^nx^n = 1 + ax + a^2x^2 + ...$$

Now, in order to solve for A(x), we must solve by partial fractions.

$$A(x) = \frac{x}{(1-x)(1-2x)} = \frac{P}{1-x} + \frac{Q}{1-2x}$$

And our goal is to solve for P and Q. Our first step will be to find a common denominator for the right hand side of the equation.

$$\frac{x}{(1-x)(1-2x)} = \frac{P(1-2X)+Q(1-x)}{(1-x)(1-2x)}$$

$$\frac{x}{(1-x)(1-2x)} = \frac{(P+Q)-x(2P+Q)}{(1-x)(1-2x)}$$

$$x = P(1-2x) + Q(1-x)$$

Solving this equation for when x=1 and x=1/2, we get:

$$1 = P(1-2) + Q(0) => P = -1$$

$$1/2 = P(0) + Q(1/2) => Q = 1$$

Now, we have:

$$A(x) = \frac{-1}{1-x} + \frac{1}{1-2x}$$

$$\sum_{n=0}^{\infty} a_n x^n = -\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} 2^n x^n$$

Just looking at the coefficients of x^n, we get:

$$a_n = 2^n - 1$$

And our original recursive relation was $a_{n+1} = 2a_n + 1$

Example: Fibonacci Numbers Let F_n be the n^{th} Fibonacci number where $F_0 = 1$ and $F_1 = 1$
Our recurrence formula is: $F_n = F_{n-1} + F_{n-2}$
Our goal is to find the generating function defined by $G(x) = \sum_{n=0}^{\infty} F_n x^n$ Our first step is to multiply both sides of the recurrence by x^n
Then we have:

$$F_n x^n = F_{n-1} x^n + F_{n-2} x^n$$

Next step: sum both sides over all values of n

$$\sum_{n=2}^{\infty} F_n x^n = \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n$$

Our next step is to add and subtract values of $F_n x^n$ so that it matches up with our definition of $G(x)$. Looking at the left hand side of our equation:

$$\sum_{n=2}^{\infty} F_n x^n + F_1 x^1 + F_0 x^0 - F_1 x^1 - F_0 x^0$$

$$\sum_{n=0}^{\infty} F_n x^n - x - 1$$

Which gives us:

$$G(x) - x - 1 = \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n$$

Looking at the right hand side of our equation:

$$\sum_{n=2}^{\infty} F_{n-2} x^n = x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2}$$

$$\sum_{n=2}^{\infty} F_{n-1} x^n = x \sum_{n=2}^{\infty} F_{n-1} x^{n-1}$$

$$x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2}$$

$$x \sum_{n=1}^{\infty} F_n x^n + x^2 \sum_{n=0}^{\infty} F_n x^n$$

Now, we have to add and subtract $F_0 x^0$ so that it matches up with our definition of G(x)

$$(x \sum_{n=1}^{\infty} F_n x^n + F_0 x^0 - F_0 x^0) + x^2 G(x)$$

$$x(G(x) - 1) + x^2 G(x)$$

$$G(x) - x + x^2 G(x)$$

Now, we are left with:

$$G(x) - x - 1 = x(G(x) - 1) + x^2 G(x)$$

$$G(x) - xG(x) + x^2 G(x) = x + 1 - x = 1$$

$$G(x)(1 - x - x^2) = 1$$

$$G(x) = \frac{1}{(1 - x - x^2)}$$

Using the quadratic formula to factor G(x), we get our two roots to be:

$$\frac{-1 + \sqrt{5}}{2} \; and \frac{-1 - \sqrt{5}}{2}$$

Next, we will define φ and ψ:

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

$$\psi = \frac{1 - \sqrt{5}}{2}$$

Which gives us:

$$x^2 - x - 1 = (x + \psi)(x + \varphi)$$

Thus:

$$G(x) = \frac{-1}{(x + \psi)(x + \varphi)}$$

Our next step is to split $G(x)$ up using partial fractions.

$$\frac{-1}{(x + \psi)(x + \varphi)} = \frac{A}{x + \psi} + \frac{B}{x + \varphi}$$

$$-1 = A(x + \varphi) + B(x + \psi)$$

Plugging in $x = -\varphi$:

$$-1 = A(0) + B(-\varphi + \psi)$$

$$-1 = B(\sqrt[-1]{5})$$

$$B = \frac{1}{\sqrt{5}}$$

Plugging in $x = -\psi$:

$$-1 = A(-\psi + \varphi) + B(0)$$

$$-1 = A(\sqrt{5})$$

$$A = \frac{-1}{\sqrt{5}}$$

$$G(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{x + \varphi} - \frac{1}{x + \psi}\right)$$

$$=> \frac{1}{\sqrt{5}} \, (\frac{\frac{1}{\varphi}}{\frac{x}{\varphi}+1} - \frac{\frac{1}{\psi}}{\frac{x}{\psi}+1})$$

$$\frac{1}{\psi} = \frac{2}{1-\sqrt{5}} * \frac{1+\sqrt{5}}{1+\sqrt{5}}$$

$$=> \frac{2(1+\sqrt{5})}{-4} \; = -\frac{1+\sqrt{5}}{2} \; = -\varphi$$

Thus, we have:

$$G(x) = \frac{1}{\sqrt{5}} \, (\frac{\frac{1}{\varphi}}{\frac{x}{\varphi}+1} \; - \frac{\frac{1}{\psi}}{\frac{x}{\psi}+1})$$

$$G(x) = \frac{1}{\sqrt{5}} \, (\frac{-\psi}{1-x\psi} + \frac{\varphi}{1-x\varphi})$$

$$G(x) = \frac{1}{\sqrt{5}} \, (\varphi \sum_{n=0}^{\infty} \varphi^n x^n - \psi \sum_{n=0}^{\infty} \psi^n x^n)$$

$$F_n x^n = \frac{1}{\sqrt{5}} \, (\varphi^{n+1} x^n - \psi^{n+1} x^n)$$

$$F_n = \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}}$$

Our closed form for the n^{th} Fibonacci number is $F_n = \frac{\varphi^{n+1}-\psi^{n+1}}{\sqrt{5}}$