Applied Combinatorics Notes 10/2/17

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Theorem There are $(n-1)!$ cycles of length n.

A cycle is a permutation where all of the elements in the permutation are part of the same "loop" when we iterate the permutation.

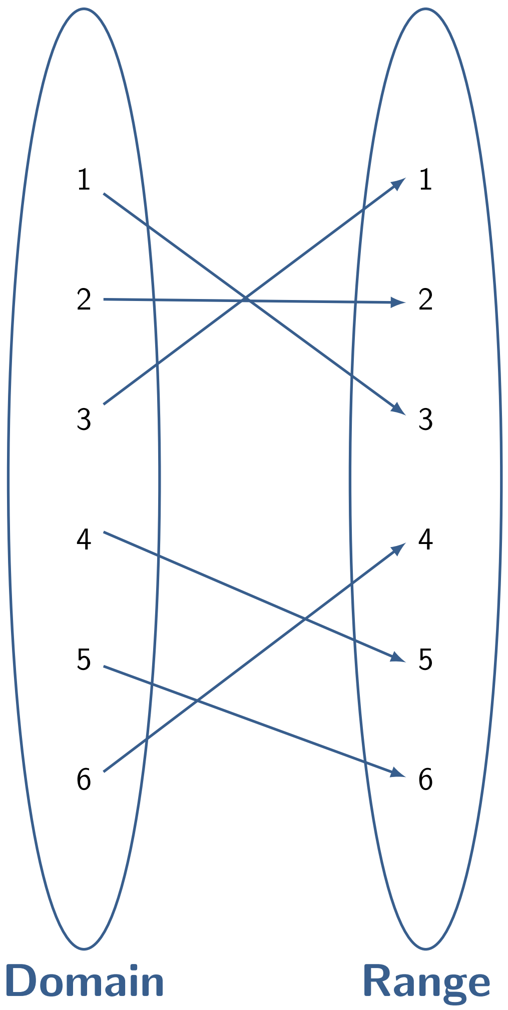
Function notation for a permutation: Let π be a permutation of n. Then $\pi : [n] \to [n]$ is a one-to-one and onto function.

Example: $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 5 & 6 & 4 \end{pmatrix}$

This is called **two line notation for a permutation**. It indicates that $\pi(1) = 3, \pi(2) = 2, \pi(3) = 1, \pi(4) = 5, \pi(5) = 6$, and $\pi(6) = 4$.

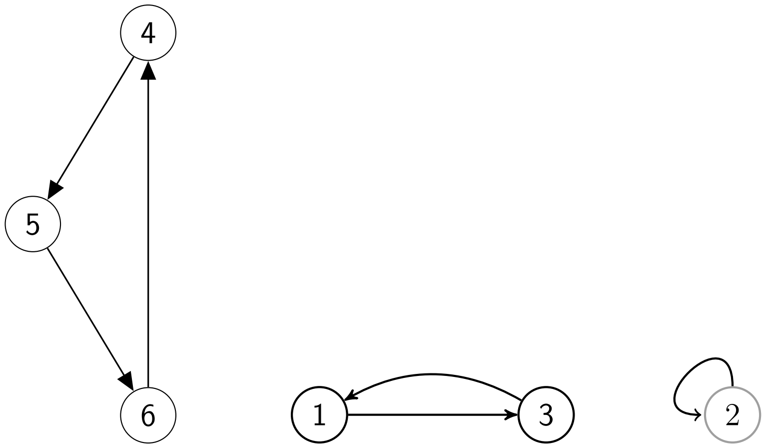
Often we omit the top line and write $\pi = \;3\;\;2\;\;1\;\;5\;\;6\;\;4$. This is called **one line notation for a permutation**.

A "function way" of looking at a permutation is given by the following drawing.



Note the domain and the range are the same set.

Another visualization for permutations can be created using one dot to represent both the input and the output.This is shown in the following diagram.



Definition A **cycle** in a permutation π is a sequence $(a_1, a_2, ..., a_k)$ such that $\pi(a_1) = a_2$, $\pi(a_2) = a_3$,...,$\pi(a_{k-1}) = a_k$,$\pi(a_k) = a_1$.
Example: If $\pi = \begin{array}{cccccc} 3 & 2 & 1 & 5 & 6 & 4 \end{array}$
Then $(4, 5, 6)$ and $(1, 3)$ are cycles.
Lemma If π is a permutation of n and $i \in [n]$ then there exists a k such that $\pi^k(i) = \pi(\pi(\pi(......\pi(i))) = i$ (apply π to i k times and it will be equivalent to i). In particular, we can choose the lease such value of k in which case, $i, \pi(i), \pi^2(i), ..., \pi^{k-1}(i)$ are all distinct.
Proof Consider $i, \pi(i), \pi^2(i), ..., \pi^n(i)$. There are n+1 numbers in this list and they all come from [n]. So, by the Pigeon-hole Principle, two of the numbers in the list must be the same. Suppose

$$\pi^a(i) = \pi^b(i) \text{ where } a < b$$

Because π is a permutation it is one-to-one and onto. This means it has an inverse function, π^{-1}. Apply π^{-1} to both sides of this equation a times.

$$(\pi^{-1})^a \circ \pi^a(i) = (\pi^{-1})^a \circ \pi^b(i) \text{ where } a < b$$
$$i = \pi^{b-a}(i)$$

This shows we found some number of times we can apply π to i and get i. So, $k = b - a$ works. Let k be the least number with $\pi^k(i) = i$. Claim $i, \pi(i), \pi^2(i), ..., \pi^{k-1}(i)$ are all distinct. If they were not distinct, then we would have

$$\pi^c(i) = \pi^d(i) \text{ where } 1 \leq c \leq d \leq k - 1$$

Using the same trick as before we can apply π^{-1} to both sides of the equation c times. This gives

$$i = \pi^{d-c}(i)$$

This contradicts the minimality of k. It follows that $i, \pi(i), \pi^2(i), ..., \pi^{k-1}(i)$ are all distinct. ■

This lemma lets us find the cycle containing i. $(i, \pi(i), \pi^2(i), ..., \pi^{k-1}(i))$ is a cycle of π containing i. In general, we call a cycle of length k a **k cycle**. Note that this notation for a cycle is not unique. We can change the element listed first.
Example: (4, 5, 6) is the same cycle as (5, 6, 4) and (6, 4, 5).

Theorem Any permutation π can be written as the disjoint composition of cycles $C_1, C_2, ..., C_k$.

$$\pi = C_1 \circ C_2 \circ C_3 \circ ..... \circ C_k$$

By disjoint we mean that if $C_a(i) \neq i$, then $C_b(i) = i$ if $a \neq b$.

Proof First note that if 2 cycles C_a and C_b are disjoint then they commute.

$$C_a \circ C_b = C_b \circ C_a$$

Any element $i \in [n]$ is affected by at most one of these cycles so the order of the composition doesn't matter.
We will prove the theorem by induction on n.
Base case: If n=1, then

$$\pi = 1$$

This permutation is a 1-cycle. Thus, the base case holds.
Induction Hypothesis: Now suppose the theorem holds for n. Every permutation on n elements can be decomposed as the product of disjoint cycles.
We want to prove that it holds for cycles of length n+1.
Let π be a permutation of $[n+1]$. Our lemma says that $(n+1)$ is part of some cycle C where

$$C = (n+1, \pi(n+1), ...)$$

Let's define a permutation $\widetilde{\pi}$ by

$$\widetilde{\pi} : [n] \to [n]$$

$$\widetilde{\pi} = \begin{cases} \pi(i) & i \notin C \\ i & i \in C \end{cases}$$

Note that $\widetilde{\pi}$ is one-to-one and onto since $n+1$ is mapped to itself.

By our induction hypothesis $\widetilde{\pi}$ is a product of disjoint cycles, i.e
$\widetilde{\pi} = C_1 \circ C_2 \circ C_3 \circ ..... \circ C_k$.
Ignore any 1-cycles. Since $\widetilde{\pi}$ sends elements of c to themselves, each of $C_1, C_2, ..., C_k$ are disjoint from C. So, $\pi = C \circ \widetilde{\pi}$ and now we can write
$\pi = C \circ C_1 \circ C_2 \circ C_3 \circ ..... \circ C_k$.
We claim that these cycles are unique. Let

$$\pi = D_1 \circ D_2 \circ .... \circ D_l, \text{ where } D_i \text{ are disjoint cycles.}$$

Pick some $i \in [n]$. Then at most one of $C_a(i) \neq i$, and at most one of $D_b(i) \neq i$. There is a unique cycle containing i by our lemma so it $D_b = C_a$.

Since disjoint cycles commute this means that the cycles $D_1, D_2, ..., D_l$ are the same as the cycles $C_1, C_2, ..., C_k$, just possibly in a different order. Thus, any permutation can be written as a composition of disjoint cycles. ■

Example: $\pi = \ 3 \ \ 2 \ \ 1 \ \ 5 \ \ 6 \ \ 4$
$\pi = (1\ 3) \circ (2) \circ (4\ 5\ 6)$