Applied Combinatorics Notes 10/16/17

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Hat-check Problem

Suppose n people all check their hats at the door. Later, everyone receives a random hat. What is the probability that no one receives their own hat?

All of the ways hats can be mixed up is just a permutation of n. So, there are $n!$ total ways to mix up the hats. To calculate the probability we can count all the permutations where no one gets their own hat and divide it by $n!$.

In the permutations where no one gets their own hat there are no fixed points. **Fixed points** in a permutation are numbers that are sent to themselves.

Definition: **Derangements** are permutations where no number is sent to itself (i.e there are no fixed points).

Example:

For $n = 2$ there are two possible permutations of n. The permutation will either be 2 1 or 1 2. In the first case, both people get their hat back. In the second case, neither gets their hat back. Thus, the permutation 2 1 represents the only way for $n = 2$ people to not receive their own hat.

Therefore, for $n = 2$ the probability that no one gets their hat back is

$$\frac{1}{2!} = \frac{1}{2}$$

Our Mission: Count the derangements of n.

To do this we will count the permutations that do have fixed points and subtract them from the total.

Let $T_i = \{$all permutations of n where i is fixed$\}$

Example1: $T_1 = \{$all permutations of n that send 1 to itself$\}$

All other elements in T_1 = will go to any of the remaining n-1 numbers, there is no restriction on them.

Example 2: $T_2 = \{$all permutations of n that send 2 to itself $\}$

Again, all other elements in T_2 = will go to any of the remaining n-1 numbers, there is no restriction on them.

Let $S_n = \{$all permutations of n$\}$

The number of derangements of n $= |S_n| - |T_1 \cup T_2 \cup .... \cup T_n|$

Note that the $T_i's$ are not disjoint so we need to use the Principle of Inclusion/Exclusion.

From last class...

$$|\bigcup_{i=1}^{n} T_i| = \sum_{S \subset [n]} (-1)^{|S|+1} \cdot |\bigcap_{i \in S} T_i|$$

We need to find the size of $|\bigcap_{i \in S} T_i|$.

For any set S, only the size of the set matters in this equation.

If $|S| = 1$: Then this counts the permutations where 1 is fixed and the the rest of the $n-1$ numbers are not fixed. So, $|\bigcap_{i \in S} T_i| = |T_i| = (n-1)!$

If $|S| = 2$: Then, this is counting the permutations that fix 1 and 2. All other $n-2$ elements are not fixed. So, $|\bigcap_{i \in S} T_i| = (n-2)!$

If $|S| = k$: This counts the permutations that send k to themselves. The rest of the $n-k$ elements are not fixed. $|\bigcap_{i \in S} T_i| = (n-k)!$

We can now plug this back into out equation for the number of derangements of n...

The number of derangements of $n = |S_n| - |T_1 \cup T_2 \cup .... \cup T_n|$

The number of derangements of $n = n! - \sum_{S \subset [n]} (-1)^{|S|+1} \cdot (n - |S|)!$

Since the summation depends on the size of S and not what is in S we can rewrite the summation using all possible sizes of the set S. When doing this we must multiply the term within the summation by $\binom{n}{k}$, because subsets of the same size each contribute to the summation.

This gives,

The number of derangements of $n = n! - \sum_{k=1}^{n} \binom{n}{k} \cdot (-1)^{k+1} \cdot (n-k)!$

Note, for this summation $k = 0$ gives $n!$. So, we can rewrite this equation...

The number of derangements of $n = -\sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^{k+1} \cdot (n-k)!$

This expression can be simplified to

The number of derangements of $n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

Back to the original question. What is the probability that no one receives their own hat?

Let p(n) equal the probability that none of the n people receive their own hat back. Then, $p(n) = \frac{total\ number\ of\ derangements\ of\ n}{total\ number\ of\ permutations\ of\ n}$ So,

$p(n) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

For n=2:

$p(2) = \sum_{k=0}^{2} \frac{(-1)^k}{k!} = \frac{1}{2}$

For n=3:

$p(3) = \sum_{k=0}^{3} \frac{(-1)^k}{k!} = \frac{1}{3}$

For n=4:

$p(n) = \sum_{k=0}^{4} \frac{(-1)^k}{k!} = \frac{3}{8}$

What happens as $n \to \infty$?

$$lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e} \approx 0.36...$$

Therefore there is $\approx \frac{1}{3}$ probability as n gets large. ■

The Ballot Problem

Two candidates - Alice and Bob - are running in an election. Suppose they each get n votes, but the votes come in one at a time. How many different orders can the votes come in so that Bob is never ahead?

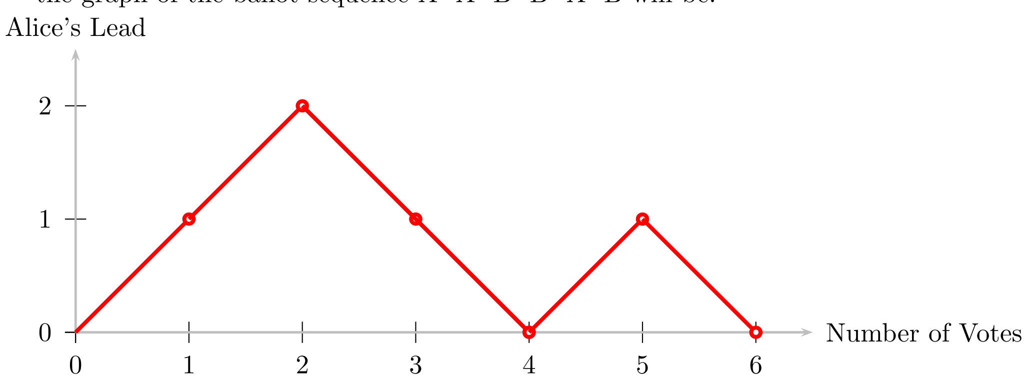
Example:
For $n = 3$ two possible ballot sequences are:
A A B B A B, and A B A B B A.

In the first ballot sequence Bob never takes the lead. This is one sequence we want to count.

In the second sequence Bob took the lead. We do not want to count these types of sequences.

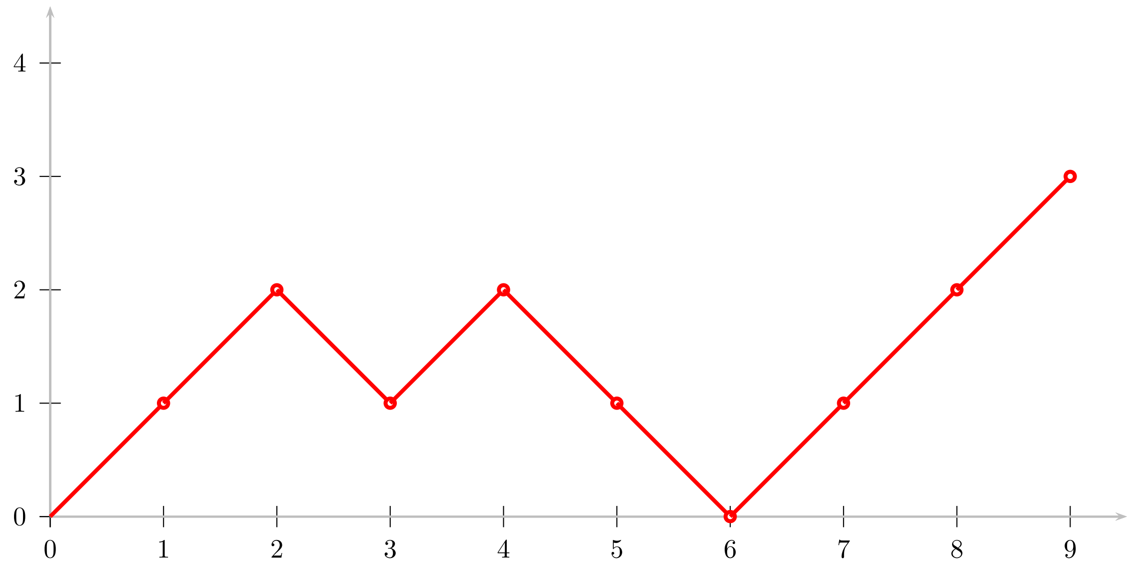
In the election we have $2n$ total ballots. In the case where we don't care who is in the lead we can choose n ballots for Alice, and the rest will be the votes for Bob. Thus the total number of possible ballot sequences (including those where Bob is in the lead) is $\binom{2n}{n}$.

Graphing the votes in the case when Bob never takes the lead. For example the graph of the ballot sequence A A B B A B will be:



This graph will never cross below the x-axis if Bob never takes the lead. As votes come in the graph can move up one and over one, or down one and over one.

Any such path that starts at (0,0) and takes steps either up (1,1) or down (1,-1) is called a **lattice path**.
Example of a lattice path of length 9.



How many lattice paths are there total of length n? There are 2^n because at each of the n steps there are two options for the next move.

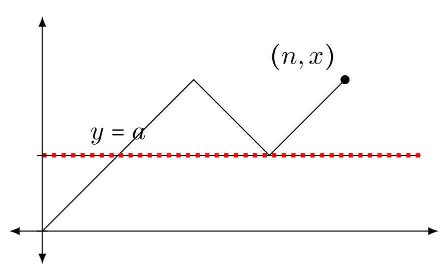
Let $C_{n,x}$ = { All lattice paths of length n that start at (0,0), and end at position (n,x) }
where n = the length of the path (total number of steps), and x = the final height of the path.

Let u = the number of up steps, and let d = the number of down steps. Then, $u + d = n$ and $u - d = x$. Thus, $u = \frac{n+x}{2}$

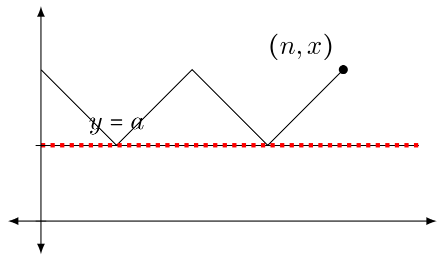
Therefore, the size of $C_{n,x}$ is equal to the way to choose the number of up steps necessary to end at (n, x) from the total number of n steps.

$$|C_{n,x}| = \binom{n}{\frac{n+x}{2}}$$

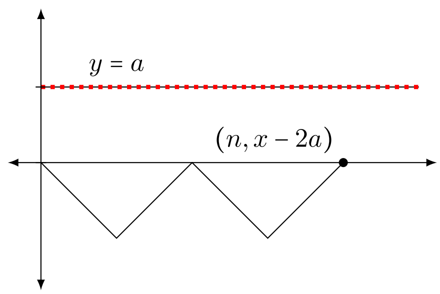
Let $C_{n,x,a}$ = { All lattice paths of length n that start at (0,0), and end at position (n,x), and that reach a height a at some point in the path}



We will use the Reflection Principle for paths reaching height a. To do this take a path that reaches height a at some point. Find the first point it reaches height a and reflect the entire path before that point above the line $y = a$ (all up steps become down steps and all down steps become up steps before that point).



This gives us a lattice path from $(2a, 0)$ to (n, x). We now shift this lattice path so that it begins at $(0, 0)$.



This path now starts at $(0, 0)$ and ends at $(n, x-2a)$. (Note: for this example $a = 1$, $x-2a$ does not always equal 0). Every step used to create this new lattice path is reversible so this creates a bijection from $C_{n,x,a}$ to $C_{n,(x-2a)}$.

So, it follows that
$|C_{n,x,a}| = |C_{n,(x-2a)}|$ for $a > x$.

To count ballot sequences we want lattice paths that stay above the x-axis but end at the point $(2n, 0)$. We will call these "good paths."

If the path reaches a height $a = -1$ then it has crossed below the x-axis. We will call these "bad paths." The number of "bad paths" from $(0, 0)$ to $(2n, 0)$ is

$$|C_{2n,0,-1}| = |C_{2n,2}| \\ = \binom{2n}{\frac{2n+2}{2}} = \binom{2n}{n+1}$$

So there are a total of $\binom{2n}{n+1}$ "bad paths." We can subtract this from the total number of paths to get the total number of "good paths."

Therefore, the total number of "good paths" is given by

$$\binom{2n}{n} - \binom{2n}{n+1} \\ = \frac{1}{n+1}\binom{2n}{n}$$

Note that this indicates the number of "good paths" is counted by the Catalan numbers. ■
Definition: The **nth Catalan number** is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$