(1) Find all first and second partial derivatives.

a.) $f(x, y) = x^2 \ln(x^2 + y^2)$

$$\frac{\partial f}{\partial x} = 2x\ln(x^2+y^2) + \frac{2x^3}{x^2+y^2} \qquad \frac{\partial f}{\partial y} = \frac{2x^2y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{10x^2}{x^2+y^2} + 2\log\left(x^2+y^2\right) - \frac{4x^4}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{4xy}{x^2+y^2} - \frac{4x^3y}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2x^2}{x^2+y^2} - \frac{4x^2y^2}{\left(x^2+y^2\right)^2}$$

b.) $g(u, v) = \dfrac{u+2v}{u^2+v^2}$

$$g_u(u,v) = \frac{1}{u^2+v^2} - \frac{2u(u+2v)}{\left(u^2+v^2\right)^2}, \qquad g_v(u,v) = \frac{2}{u^2+v^2} - \frac{2v(u+2v)}{\left(u^2+v^2\right)^2}$$

$$g_{uu}(u,v) = \frac{8u^2(u+2v)}{\left(u^2+v^2\right)^3} - \frac{4u}{\left(u^2+v^2\right)^2} - \frac{2(u+2v)}{\left(u^2+v^2\right)^2}$$

$$g_{uv}(u,v) = g_{vu}(u,v) = -\frac{4u}{\left(u^2+v^2\right)^2} + \frac{8uv(u+2v)}{\left(u^2+v^2\right)^3} - \frac{2v}{\left(u^2+v^2\right)^2}$$

$$g_{vv}(u,v) = \frac{8v^2(u+2v)}{\left(u^2+v^2\right)^3} - \frac{8v}{\left(u^2+v^2\right)^2} - \frac{2(u+2v)}{\left(u^2+v^2\right)^2}$$

(2) Find an equation of the tangent plane to the surface $z = e^x \cos y$ at the point $(0, 0, 1)$.

$z = x + 1$

(3) Find the equation of the tangent plane to the level surface $7 = f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3, 2, 6)$.

$$\frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6) = 0$$

(4) Show that the following limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^6}{x^4+y^{12}}.$$

Along the line $x = 0$, the limit becomes

$$\lim_{y\to 0} \frac{0}{0+y^6} = 0.$$

Along the line $x = y^3$ the limit becomes

$$\lim_{y\to 0} \frac{y^{12}}{y^{12}+y^{12}} = \lim_{y->0} \frac{1}{2} = \frac{1}{2}.$$

Since these two values are not equal, the limit does not exist.

(5) Let C be the curve with parametrization $\mathbf{r}(t) = t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j}$. Find the equation of the tangent line to the curve C at $t = \frac{\pi}{4}$.

$$\mathbf{r}'(t) = (t\cos t + \sin t)\,\mathbf{i} + (-t\sin t + \cos t)\,\mathbf{j}$$

so the tangent line is parameterized by

$$x(s) = \frac{\pi}{4\sqrt{2}} + s\left(\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$y(s) = \frac{\pi}{4\sqrt{2}} + s\left(-\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

(6) Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2, 1, 3)$. You don't have an equation for S, but you know that the curves

$$r_1(t) = \langle 2 + 3t,\, 1 - t^2,\, 3 - 4t + t^2 \rangle, \quad \text{and}$$

$$r_2(u) = \langle 1 + u^2,\, 2u^3 - 1,\, 2u + 1 \rangle$$

both lie on S and pass through P. Find an equation of the tangent plane $T_P S$. $r_1(t)$ passes through (2,1,3) when $t = 0$, and $r_2(t)$ passes through (2,1,3) when $u = 1$, so we take the tangent vectors at each of these curves at this point, we get $r_1'(0) = \langle 3, 0 - 4\rangle$, $r_2'(1) = \langle 2, 6, 2\rangle$. Since both of these tangent vectors lie in the tangent plane at this point, their cross product $\langle -14, 18\rangle$ is normal to the plane at this point, thus an equation for the tangent plane is

$$0 = 24(x - 2) - 14(y - 1) + 18(z - 3).$$

(7) Find the directional derivative of $f(x, y) = x^2 e^{-y}$ at the point $(-2, 0)$, in the direction of the point $(2, -3)$. The vector pointing from (-2,0) to (2,-3) is $\langle 4, -3\rangle$. The unit vector pointing in this direction is $u = \langle \frac{4}{5}, -\frac{3}{5}\rangle$. The gradient of $f(x, y)$ at (-2,0) is $\nabla f(-2, 0) = \langle -4, -4\rangle$, so

$$D_u f(-2, 0) = \langle -4, -4\rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle = \frac{-16 + 12}{5} = -\frac{4}{5}.$$

(8) *a.*) Find the gradient of f; *b.*) Evaluate the gradient at the point P; *c.*) Find the rate of change of f at the point P in the direction of the vector $\mathbf{u}$.

$$f(x, y) = \sin(2x + 3y)$$

$$P = (-6, 4)$$

$$\mathbf{u} = \frac{1}{2}(\sqrt{3}\mathbf{i} - \mathbf{j})$$

a.$\nabla f = \langle 2\cos(2x + 3y), 3\cos(2x + 3y)\rangle$.
b. $\langle 2, 3\rangle$.
c. $\sqrt{3} - \frac{3}{2}$

(9) Find the maximum rate of change of the function $f(x, y) = y\,e^{xy}$ at the point $(0, 2)$, and give the direction that it occurs. Since $\nabla f = \langle y^2 e^{xy}, (xy + 1)\,e^{xy}\rangle$, the greatest rate of change is $\sqrt{4^2 + 1^2} = \sqrt{17}$, which occurs in the direction $\left\langle \frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \right\rangle$.

(10) (a) Find the unique critical point of the function

$$f(x,y) = x^2 + 3xy + 2y^2 - 8x - 11y + 30.$$

(b) Is this critical point a minimum, maximum, or saddle?

Since $f_x = 2x + 3y - 8$ and $f_y = 3x + 4y - 11$, the only critical point is $(x,y) = (1,2)$. Since $D = 2 \cdot 4 - 3^2 = -1$ this is a saddle point.

(11) Find the absolute maximum and the absolute minimum of

$$f(x,y) = (x-1)(y-2)$$

in the closed triangle $0 \le x$, $0 \le y$, $x + y \le 7$ bounded by the x-axis, the y-axis, and the line $x + y = 7$.

The only critical point of this function is $(1,2)$, which is inside of this triangle, and it has no critical points along the x or y axes. Along the line $x + y = 7$, the function has a critical point at (3,4), so we need to check the points

(x,y)	$f(x,y)$
(1,2)	0
(3,4)	4
(0,0)	2
(7,0)	-12
(0,7)	-5

Thus the absolute maximum is 4 and the absolute minimum is -12.

(12) Find parametric equations for

(a) The plane through $(1,3,4)$ and orthogonal to $\mathbf{n} = \langle 2, 1, -1 \rangle$.
$r(u,v) = \langle u, v, 2(u-1) + (v-3) + 4 \rangle$.

(b) The sphere centered at the origin and having radius 5.
$r(u,v) = \langle 5\cos u \sin v, 5 \sin u \sin v, 5 \cos v \rangle$, $u \le 2\pi$, $0 \le v \le \pi$.

(c) The sphere centered at the point $(2,-1,3)$ and with radius 5.
$r(u,v) = \langle 5\cos u \sin v - 2, 5 \sin u \sin v + 1, 5 \cos v - 3 \rangle$, $u \le 2\pi$, $0 \le v \le \pi$.

(d) The cone $x^2 + y^2 = z^2$. $0 \le z \le 2$,
$r(u,v) = \langle v \cos u, v \sin u, v \rangle$, $0 \le u \le 2\pi$, $0 \le v \le 2$.

(13) Evaluate $\displaystyle\int_{x=0}^{1} \int_{y=2x}^{1} \int_{z=x^3+y}^{x^2+2y} y \, dz \, dy \, dx$. $-\frac{43}{120}$

(14) Let $\mathbf{F}(x,y) = x^2 y \mathbf{i} + 2xy^2 \mathbf{j}$. Compute

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Where C is the line from $(0,0)$ to $(2,4)$ along the curve $y = x^2$.

Parameterize C by $\mathbf{r(t)} = \langle \mathbf{t}, \mathbf{t^2} \rangle$, $0 \le t \le 2$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \langle t^4, 2t^5 \rangle \cdot \langle 1, 2t \rangle \, dt = \int_0^2 t^4 + 4t^6 \, dt = \frac{2^5}{5} + \frac{2^9}{7}$$

(15) Evaluate the integral $\displaystyle\int\int_R x \, dx \, dy$ where R is the triangle with vertices $(1,2)$, $(3,3)$, $(4,5)$.

$$\int_2^3 \int_{y-1}^{2y-3} x \, dx \, dy + \int_3^5 \int_{y-1}^{\frac{y}{2}+\frac{3}{2}} x \, dx \, dy = 4$$

(16) Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = (x^2+y^2)\mathbf{i} + 2xy\mathbf{j}$ and Let γ be the curve which follows the parabola $y = x^2$ from $(0,0)$ to $(2,4)$, and the line segments from $(2,4)$ to $(0,4)$, and from $(0,4)$ to $(0,0)$. Use Green's theorem to evaluate $\int_\gamma \mathbf{F} \cdot \mathbf{dr}$.

$$\int_0^2 \int_{x^2}^4 2y - 2y\, dydx = 0$$

(17) The cardioid is the curve with polar equation

$$r = 1 - \cos\theta, \qquad 0 \le \theta \le 2\pi.$$

and is given parametrically by $x(t) = \cos(t) - \cos^2(t)$, $y(t) = \sin(t) - \cos(t)\sin(t)$, $0 \le t \le 2\pi$.

Use Green's theorem to find the area of the region inside the cardioid by evaluating the integral

$$\oint_C x\, dy.$$

$$\begin{aligned}
\oint_C x\, dy &= \int_0^{2\pi} (\cos t - \cos^2 t)(\cos t - \cos^2 t + \sin^2 t)dt \\
&= \int_0^{2\pi} \cos^2 t - 2\cos^3 t + \cos^4 t + \cos t \sin^2 t - \cos^2 t \sin^2 t dt \\
&= \int_0^{2\pi} -2\cos t + 3\cos t \sin^2 t + 2\cos^4 t\, dt \\
&= \int_0^{2\pi} -2\cos t + 3\cos t \sin^2 t\, dt + \int_0^{2\pi} \frac{1 + 2\cos 2t + \cos^2 2t}{2}\, dt \\
&= 0 + \int_0^{2\pi} \frac{1 + 2\cos 2t}{2} + \frac{1 + \cos 4t}{4} = \frac{3\pi}{2}\, dt
\end{aligned}$$

(18) Let S be the quarter of the disk of radius 1 in the yz-plane centered at the origin for which $y \ge 0$ and $z \ge 0$. Consider the vector field $\mathbf{F}(x,y,z) = \langle y, z, x\rangle$. Orient S so that its normal vector points in the direction of the positive x-axis.

(a) Give the boundary C of S the orientation induced by the right-hand rule. With this orientation, compute the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{s}.$$

Break this curve into three pieces:
C_1: $r_1(t) = \langle 0, t, 0\rangle$, $(0 \le t \le 1)$
C_2: $r_2(t) = \langle 0, \cos t, \sin t\rangle$, $(0 \le t \le \pi/2)$ and
C_3: $r_3(t) = \langle 0, 0, 1-t\rangle$, $(0 \le t \le 1)$.

Then

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \int_{C_1}\mathbf{F}\cdot d\mathbf{s} + \int_{C_2}\mathbf{F}\cdot d\mathbf{s} + \int_{C_3}\mathbf{F}\cdot d\mathbf{s}$$
$$= \int_0^1 \langle t,0,0\rangle\cdot\langle 0,1,0\rangle dt + \int_0^{\frac{\pi}{2}} \langle \cos t, \sin t, 0\rangle\cdot\langle 0,-\sin t,\cos t\rangle dt + \int_0^1 \langle 0,1-t,0\rangle\cdot\langle 0,0,-1\rangle dt$$
$$= 0 + \int_0^{\pi/2} -\sin^2 t\, dt + 0 = -\frac{\pi}{4}$$

(b) Compute the curl of the vector field, $\nabla\times\mathbf{F}$. $\langle -1,-1,-1\rangle$

(c) Verify Stokes' Theorem by computing

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}.$$

Since this region lies in the yz plane, it has normal vector $\langle 1,0,0\rangle$. Thus

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_S \langle -1,-1,-1\rangle\cdot\langle 1,0,0\rangle dS = \iint_S -1 dS = -\frac{\pi}{4}$$

(19) Let $W(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ and let $\mathbf{F} = \nabla W$ be the gradient of W.

(a) Calculate $\mathbf{F}$ and the divergence of $\mathbf{F}$, $\nabla\cdot\nabla W$.

$$\mathbf{F} = \left\langle -\frac{x}{(x^2+y^2+z^2)^{3/2}}, -\frac{y}{(x^2+y^2+z^2)^{3/2}}, -\frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$
$$\nabla\cdot\mathbf{F} = 0$$

(b) Use the divergence theorem to calculate the outward flux

$$\iint_\Sigma \mathbf{F}\cdot\, dS$$

through the surface σ which is the boundary of the solid S bounded by the xy-plane and by the hemispheres

$$z = \sqrt{4-x^2-y^2} \quad \text{and} \quad z = \sqrt{9-x^2-y^2}.$$

$$\iint_\Sigma \mathbf{F}\cdot\, dS = \iiint_S \operatorname{div}\mathbf{F}\, dV = 0$$