

Math 273 Fall 2015

Calculus I

Midterm Exam 1

Wednesday September 23 6:00-8:00 PM

Your name (please print): _____

Instructions: This is a closed book exam. You may not refer to the textbook, your notes or any other material, including a calculator. Answers do not need to be simplified, however you should **provide enough work to explain how you arrived at your answer.**

The academic integrity policy requires that you neither give nor receive any aid on this exam.

For grader use only:

Problem	Points	Score
1	30	
2	8	
3	15	
4	10	
5	6	
6	14	
7	6	
8	5	
Total	94	

1. Evaluate the following limits: (If the limit is $\pm\infty$, say so, if it does not exist, write DNE)

a. $\lim_{x \rightarrow 5} \frac{\frac{2}{x} - \frac{2}{5}}{x - 5}$

$$\lim_{x \rightarrow 5} \frac{\frac{2}{x} - \frac{2}{5}}{x - 5} = \lim_{x \rightarrow 5} \frac{\frac{10-2x}{5x}}{x - 5} = \lim_{x \rightarrow 5} \frac{-2}{5x} = -\frac{2}{25}$$

b. $\lim_{x \rightarrow -5} \frac{3x + 15}{|x + 5|}$

$$\lim_{x \rightarrow -5} \frac{3x + 15}{|x + 5|} = 3 \lim_{x \rightarrow -5} \frac{x + 5}{|x + 5|} \quad \text{DNE}$$

Because

$$\frac{x + 5}{|x + 5|} = \begin{cases} 1 & x > -5 \\ -1 & x < -5 \end{cases}$$

c. $\lim_{x \rightarrow 0} \frac{1}{x} - \frac{1}{x^2}$

$$\lim_{x \rightarrow 0} \frac{1}{x} - \frac{1}{x^2} = \lim_{x \rightarrow 0} \frac{x - 1}{x^2} = -\infty$$

Since $x - 1 \rightarrow -1$ while $x^2 \rightarrow 0$.

d. $\lim_{x \rightarrow \infty} \frac{2 + 3x + 5x^2}{3 - 2x^2}$

$$\lim_{x \rightarrow \infty} \frac{2 + 3x + 5x^2}{3 - 2x^2} = \lim_{x \rightarrow \infty} \frac{2 + 3x + 5x^2}{3 - 2x^2} \left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right) = \lim_{x \rightarrow \infty} \frac{\frac{2}{x^2} + \frac{3}{x} + 5}{\frac{3}{x^2} - 2} = -\frac{5}{2}$$

e. $\lim_{x \rightarrow 1} \frac{3 - 4x + x^2}{x - 1}$

$$\lim_{x \rightarrow 1} \frac{3 - 4x + x^2}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 3)(x - 1)}{x - 1} = \lim_{x \rightarrow 1} x - 3 = -2$$

e. $\lim_{x \rightarrow 1} 10^{f(x)}$ where $f(x) = \begin{cases} 2x & x < 1 \\ 0 & x = 1 \\ 3 - x & x > 1 \end{cases}$

10^x is continuous everywhere, so

$$\lim_{x \rightarrow 1} 10^{f(x)} = 10^{\lim_{x \rightarrow 1} f(x)}$$

Looking at the definition of $f(x)$, we see that both

$$\lim_{x \rightarrow 1^+} f(x) = 2$$

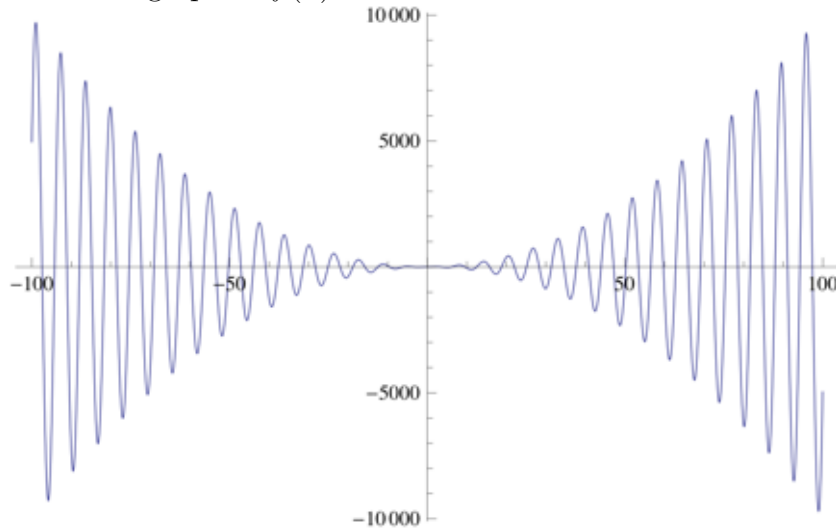
$$\lim_{x \rightarrow 1^-} f(x) = 2,$$

so $\lim_{x \rightarrow 1} 10^{f(x)} = 10^2 = 100$.

2. Define

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

a. Draw the graph of $f(x)$.



b. Where is $f(x)$ continuous? Use the limit definition of continuity to explain your answer.

$\sin(x)$ is continuous everywhere, and $\frac{1}{x}$ is continuous everywhere except at 0, so $f(x)$ is continuous everywhere, except possibly at 0. By the squeeze theorem, using

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

we see that $\lim_{x \rightarrow 0} f(x) = 0$. Since $f(0) = 0$, the function is equal to the value of its limit, so by the definition of continuity, $f(x)$ is continuous at 0 as well.

3. Use the limit definition of the derivative to find the derivatives of the following functions. (Note: No credit will be given for using the power rule, it's not on this exam!)

a. $f(x) = 2x^2 + x$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{2(x+h)^2 + (x+h) - (2x^2 + x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2x^2 + 4xh + 2h^2 + (x+h) - (2x^2 + x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2x^2 + 4xh + 2h^2 + x + h - 2x^2 - x}{h} \\ &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 + h}{h} = \lim_{h \rightarrow 0} 4x + 2h + 1 = \boxed{4x+1} \end{aligned}$$

b. $f(x) = \frac{1}{x+2}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x+2 - (x+2+h)}{(x+2)(x+2+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{-h}{(x+2)(x+2+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x+2)(x+2+h)} = \boxed{-\frac{1}{(x+2)^2}} \end{aligned}$$

c. $f(x) = \sqrt{x}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}} \end{aligned}$$

4. State the precise $(\epsilon - \delta)$ definition of the limit, and use it to prove that

$$\lim_{x \rightarrow 3} -2x + 1 = -5.$$

The limit of $-2x + 1$ as x approaches 3 equals -5 if, for every $\epsilon > 0$, there exists a $\delta > 0$, such that if $|x - 3| < \delta$, then $|(-2x + 1) - (-5)| < \epsilon$.

Take $\delta = \epsilon/2$. Then if $|x - 3| < \delta = \epsilon/2$, we have that $2|x - 3| < \epsilon$, so

$$|-2x + 6| = |-2x + 1 - (-5)| < \epsilon.$$

5. Prove that there exists a value of x for which the function $f(x) = -100\cos(\pi x) + 10^x$ is equal to zero.
 $-100\cos(\pi x) + 10^x$ is continuous everywhere. Since $f(0) = -100 + 1 = -99$ and $f(1) = 100 + 10 = 110$, there exists a value of x between 0 and 1 where $f(x) = 0$ by the intermediate value theorem.

6. According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the Pressure P and the volume V is a constant. Suppose that for a certain gas $PV = 1000$ in \times pounds. Where pressure is measured in pounds per square inch, and volume is measured in cubic inches.

a Find the average rate of change in P as V increases from 200 in^3 to 250 in^3 .
 When V is 200 in^3 , P is 5 pounds/in^2 . When V is 250 in^3 , P is 4 pounds/in^2 , so the average rate of change is $\frac{\Delta P}{\Delta V} = \frac{4-5}{250-200} = \frac{-1}{50}$

b Express V as a function of P , and find a function for the instantaneous rate of change of V with respect to P .

$$V(P) = \frac{1000}{P}.$$

$$\begin{aligned} V'(P) &= \lim_{h \rightarrow 0} \frac{\frac{1000}{P+h} - \frac{1000}{P}}{h} = 1000 \lim_{h \rightarrow 0} \frac{\frac{P-(P+h)}{P(P+h)}}{h} \\ &= 1000 \lim_{h \rightarrow 0} \frac{\frac{-h}{P(P+h)}}{h} = 1000 \lim_{h \rightarrow 0} \frac{-1}{P(P+h)} = \boxed{-\frac{1000}{P^2}} \end{aligned}$$

c Find a tangent line to the function when $P=50$ pounds/in².

$$V'(50) = -\frac{1000}{2500} = -0.4.$$

So the tangent line is $V = -0.4P + c$ for some c . Plugging in the point $P=50$, $V=20$, we find $c = 40$. So the tangent line is $V = 0.4P + 40$.

7. Find all vertical and horizontal asymptotes of the function

$$f(x) = \frac{3x + 8}{x + 3} + \tan^{-1}(x)$$

$\frac{3x+8}{x+3}$ has a vertical asymptote at -3 , so $f(x)$ does as well, this is the only vertical asymptote. To find the horizontal asymptotes, we take

$$\lim_{x \rightarrow \infty} \frac{3x + 8}{x + 3} + \tan^{-1}(x) = \lim_{x \rightarrow \infty} \frac{3 + 8/x}{1 + 3/x} + \lim_{x \rightarrow \infty} \tan^{-1}(x) = 3 + \pi/2$$

and

$$\lim_{x \rightarrow -\infty} \frac{3x + 8}{x + 3} + \tan^{-1}(x) = \lim_{x \rightarrow -\infty} \frac{3 + 8/x}{1 + 3/x} + \lim_{x \rightarrow -\infty} \tan^{-1}(x) = 3 - \pi/2$$

so the horizontal asymptotes are $3 + \pi/2$ and $3 - \pi/2$.

8. Draw the derivative of the following function.

