Math 273 Fall 2015

Calculus I

Midterm Exam 3

Monday November 23 6:00-8:00 PM

Your name (please print): ________________________________

Instructions: This is a closed book exam. You may not refer to the textbook, your notes or any other material, including a calculator. Answers do not need to be simplified, however you should **provide enough work to explain how you arrived at your answer.**

The academic integrity policy requires that you neither give nor receive any aid on this exam.

For grader use only:

Problem	Points	Score
1	30	
2	15	
3	8	
4	6	
5	15	
6	8	
7	10	
EC	**	
Total	92	

1. Compute the following limits.

   a. $\lim_{x \to 5} \frac{x-5}{x^2-25}$

   $\lim_{x \to 5} \frac{x-5}{x^2-25}$ has the indeteriminate form $\frac{0}{0}$, so we apply L'Hospital's rule:

$$\lim_{x \to 5} \frac{x-5}{x^2-25} = \lim_{x \to 5} \frac{1}{2x} = \frac{1}{10}$$

   b. $\lim_{x \to 0^+} x^{1/\ln x}$

   This has the indeterminate form 0^0. Since

$$x^{1/\ln x} = e^{\ln x^{1/\ln x}} = e^{\ln x/\ln x} = e^1 = e$$

   we get that

$$\lim_{x \to 0^+} x^{1/\ln x} = e.$$

   c. $\lim_{x \to -\infty} xe^x$

   This has the indeterminate form $-\infty \cdot 0$. Since $xe^x = \frac{x}{e^{-x}}$, we can use L'Hospital's rule on this fraction and get

$$\lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = 0$$

d. $\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{\sin x + \cos x}$

By direct substitution,

$$\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{\sin x + \cos x} = \frac{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}} = 0.$$

e. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

This has the indeterminate form $\infty - \infty$. Rewriting as a common fraction,

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\sin x - x}{x \sin x}.$$

Applying L'Hospital's rule,

$$\lim_{x\to 0} \frac{\sin x - x}{x \sin x} = \lim_{x\to 0} \frac{\cos x - 1}{x\cos x + \sin x},$$

Which is still in the indeterminate form $\frac{0}{0}$, so applying L'Hopital's rule a second time,

$$\lim_{x\to 0} \frac{\cos x - 1}{x\cos x + \sin x} = \lim_{x\to 0} \frac{-\sin x}{-x\sin x + \cos x + \cos x} = \frac{0}{2} = 0.$$

f. $\lim_{x\to 0} \frac{x^2 e^x}{\cos x - 1}$.

This has the indeterminate form $\frac{0}{0}$. Applying L'Hospital's rule,

$$\lim_{x\to 0} \frac{x^2 e^x}{\cos x - 1} = \lim_{x\to 0} \frac{x^2 e^x + 2xe^x}{-\sin x},$$

Which is still in the indeterminate form $\frac{0}{0}$, so applying L'Hopital's rule a second time,

$$\lim_{x\to 0} \frac{x^2 e^x + 2xe^x}{-\sin x} = \lim_{x\to 0} \frac{x^2 e^x + 2xe^x + 2xe^x + 2e^x}{-\cos x} = \frac{2}{-1} = -2.$$

2. Let $f(x) = x(x-4)^3$.
Hint: Don't foil this out before taking derivatives! It will make it easier to solve later...

   (a) Find the intervals on which f is increasing or decreasing.
   $f'(x) = (x-4)^3 + 3x(x-4)^2$.
   Setting this equal to zero, $(x-4)^3 + 3x(x-4)^2 = 0$, we find that either $x = 4$ or else we can divide by $(x-4)^2$, so $(x-4) + 3x = 0$ and $x = 1$.
   Thus the critical values are $x = 1, 4$. Using the first derivative test we find that $f'(x)$ is negative for $x < 1$, positive for $1 < x < 4$ and also positive for $4 < x$. Thus $f(x)$ is decreasing on $(-\infty, 1)$ and increasing on $(1, 4) \cup (4, \infty)$.

   (b) Find the local maxima and minima of f. Checking the critical values $x = 1, 4$ we find that $x = 1$ is a minimum, and $x = 4$ is neither a minimum or a maximum.

   (c) Find the intervals on which f is concave up or concave down and the inflection points of f.
   $f''(x) = 3(x-4)^2 + 3(x-4)^2 + 6x(x-4)$.
   Setting this equal to zero, $3(x-4)^2 + 3(x-4)^2 + 6x(x-4) = 0$
   we get that either $x = 4$, or we can divide through by $x-4$, and get $6(x-4)+6x = 12x - 24 = 0$. So the inflection points are $x = 2, 4$. Since $f''(x)$ is positive for $x < 2$, negative for $2 < x < 4$ and positive for $x > 4$ we find that $f(x)$ is concave up on $(-\infty, 2) \cup (4, \infty)$, and concave down on $(2, 4)$.

   (d) Find the intercepts of f, and any horizontal or vertical asymptotes.
   x-intercepts: (0,0) and (4,0), y-intercept (0,0). No horizontal or vertical asympotes.

   (e) Sketch the curve.

3. Find the absolute maximum and minimum of $f(x) = 2\cos x + \sin(2x)$ on the interval $[0, \frac{\pi}{2}]$.
$f'(x) = -2\sin x + 2\cos(2x)$. Setting this equal to zero, we want to find the value of x where $\cos(2x) = \sin(x)$ for $x \in [0, \frac{\pi}{2}]$. This is true at $x = \frac{\pi}{6}$, since $\cos\frac{\pi}{3} = \sin\frac{\pi}{6} = \frac{1}{2}$. We now compute

$$f(2) = 2$$

$$f\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} > 2$$

$$f\left(\frac{\pi}{2}\right) = 0.$$

So the absolute maximum is at $x = \frac{\pi}{6}$ and the absolute minimum is at $x = \frac{\pi}{2}$.

4. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible. Let the rectangle have length l and width w. Then if the perimiter is 100 we have $2l + 2w = 100$ so $l = 50 - w$. The area of the rectangle is $A = l \times w = w(50 - w)$. Taking the derivative of this with respect to w and setting equal to zero we get that

$$-w + (50 - w) = 0$$

so $w = 25$ and thus $l = 25$ as well. This is a maximum by the second derivative test since the second derivative with respect to w, is $A''(w) = -2$ which is negative.

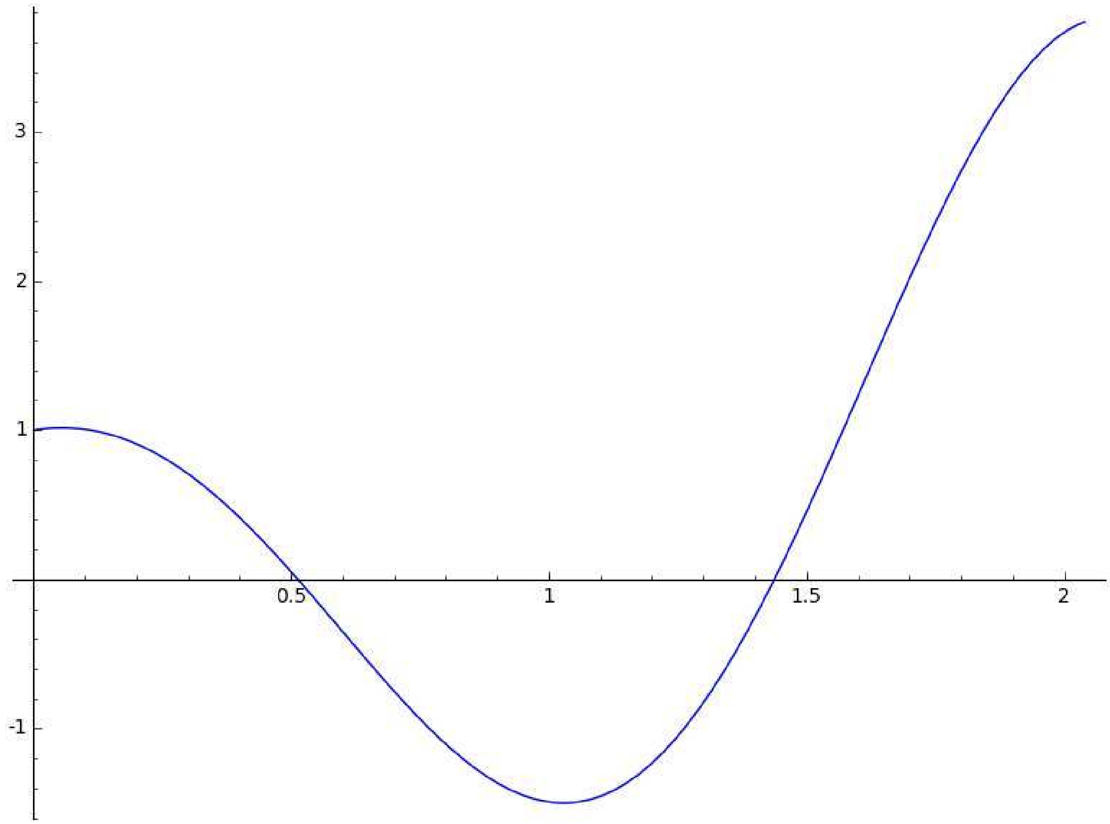
5. In each case, use the information provided to find the function $f(x)$, include a constants if necessary.

   a. $f'(x) = 2e^x + \sec^2 x$
   $f(x) = 2e^x + \tan x + C$

   b. $f''(x) = \dfrac{1}{x^2} + \dfrac{1}{2}$
   $f'(x) = -\dfrac{1}{x} + \dfrac{x}{2} + C$
   $f(x) = -\ln|x| + \dfrac{x^2}{4} + Cx + D$

   c. $f''(x) = e^x - 2\sin(x),\ \ f(0) = 3,\ \ f\left(\dfrac{\pi}{2}\right) = 0.$
   $f'(x) = e^x - 2\cos(x) + C$
   $f(x) = e^x + 2\sin(x) + Cx + D$
   $3 = f(0) = 1 + D \rightarrow D = 2.$
   $0 = f\left(\frac{\pi}{2}\right) = e^{\pi/2} + 2 + C\frac{\pi}{2} + 2 \rightarrow C = -\frac{e^{\pi/2}+2}{\pi}$
   $f(x) = e^x + 2\sin(x) - \frac{e^{\pi/2}+2}{\pi}x + 2$

6. The graph in the plot below has roots near 0.51 and 1.44. Suppose you were to use Newton's method to approximate one of these roots. For each of the values of x_0, an initial guess, make an estimate for the value of x_1, the estimate obtained by applying Newton's method once to x_0. Also state which of the two roots (0.51 or 1.44) it would eventually converge to.



(a) $x_0 = 0.6$.
$x_1 \approx 0.5$, Converges to 0.51

(b) $x_0 = 1.2$
$x_1 \approx 1.6$, Converges to 1.44

(c) $x_0 = 0.15$
$x_1 \approx 1.2$, Converges to 1.44

(d) $x_0 = 2$
$x_1 \approx 0.4$, Converges to 0.51

7. A piece of wire 10m long is cut into two pieces. One is bent into a square and one is bent into a circle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) a minimum? A square with side length w has perimeter $4w$, and a circle of radius r has circumference $2\pi r$. Thus we know that $4w + 2\pi r = 10$. Or $w = \frac{5-\pi r}{2}$.

The total area of these shapes is

$$A = w^2 + \pi r^2 = \left(\frac{5-\pi r}{2}\right)^2 + \pi r^2.$$

Taking a derivative we get that

$$A'(r) = 2\left(\frac{5-\pi r}{2}\right)\frac{-\pi}{2} + 2\pi r$$

Setting this equal to 0 we get

$$\left(\frac{5-\pi r}{2}\right) = 2r$$

so that

$$r = \frac{5/2}{2+\pi/2} = \frac{5}{4+\pi}$$

so the only critical value is $r = \frac{5}{4+\pi}$. We now want to find the absolute maximum and minimum values of the area. The two extremes are that we use none of the wire on the circle, $r = 0$ or we use all of it, $r = \frac{10}{2\pi}$. Using the second derivative test we see that $r = \frac{5}{4+\pi}$ is a minimum, and so we only need to check the endpoints

$$A(0) = 2.5^2 = 6.25$$

and

$$A\left(\frac{10}{2\pi}\right) = \pi\left(\frac{10}{2\pi}\right)^2 = \frac{100}{4\pi} = \frac{25}{\pi} > 6.25$$

So the maximum occurs when all of the length is used to make the circle.

Extra Credit: Find an antiderivative for $\ln x$. (Show work on back.) $x\ln x - x$