Math 273 Fall 2015

Calculus I

Midterm Exam 1

Wednesday September 23 6:00-8:00 PM

Your name (please print): ______________________________

Instructions: This is a closed book exam. You may not refer to the textbook, your notes or any other material, including a calculator. Answers do not need to be simplified, however you should **provide enough work to explain how you arrived at your answer.**

The academic integrity policy requires that you neither give nor receive any aid on this exam.

For grader use only:

Problem	Points	Score
1	30	
2	8	
3	15	
4	10	
5	6	
6	14	
7	6	
8	5	
Total	94	

1. Evaluate the following limits: (If the limit is $\pm\infty$, say so, if it does not exist, write DNE)

   a. $\lim\limits_{x\to 5} \dfrac{\frac{2}{x}-\frac{2}{5}}{x-5}$

$$\lim_{x\to 5} \frac{\frac{2}{x}-\frac{2}{5}}{x-5} = \lim_{x\to 5} \frac{\frac{10-2x}{5x}}{x-5} = \lim_{x\to 5} \frac{-2}{5x} = -\frac{2}{25}$$

   b. $\lim\limits_{x\to -5} \dfrac{3x+15}{|x+5|}$

$$\lim_{x\to -5} \frac{3x+15}{|x+5|} = 3 \lim_{x\to -5} \frac{x+5}{|x+5|} \qquad \text{DNE}$$

Because

$$\frac{x+5}{|x+5|} = \begin{cases} 1 & x > -5 \\ -1 & x < -5 \end{cases}$$

   c. $\lim\limits_{x\to 0} \dfrac{1}{x} - \dfrac{1}{x^2}$

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{x^2} = \lim_{x\to 0} \frac{x-1}{x^2} = -\infty$$

Since $x-1 \to -1$ while $x^2 \to 0$.

d. $\lim\limits_{x\to\infty} \quad \dfrac{2+3x+5x^2}{3-2x^2}$

$$\lim_{x\to\infty} \quad \frac{2+3x+5x^2}{3-2x^2} = \lim_{x\to\infty} \frac{2+3x+5x^2}{3-2x^2}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \lim_{x\to\infty} \quad \frac{\cancel{\frac{2}{x^2}} + \cancel{\frac{3}{x}} + 5}{\cancel{\frac{3}{x^2}} - 2} = -\frac{5}{2}$$

e. $\lim\limits_{x\to 1} \quad \dfrac{3-4x+x^2}{x-1}$

$$\lim_{x\to 1} \quad \frac{3-4x+x^2}{x-1} = \lim_{x\to 1} \frac{(x-3)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1} x-3 = -2$$

e. $\lim\limits_{x\to 1} \quad 10^{f(x)} \qquad$ where $f(x) = \begin{cases} 2x & x<1 \\ 0 & x=1 \\ 3-x & x>1 \end{cases}$

10^x is continuous everywhere, so

$$\lim_{x\to 1} 10^{f(x)} = 10^{\lim_{x\to 1} f(x)}$$

Looking at the definition of $f(x)$, we see that both

$$\lim_{x\to 1^+} f(x) = 2$$

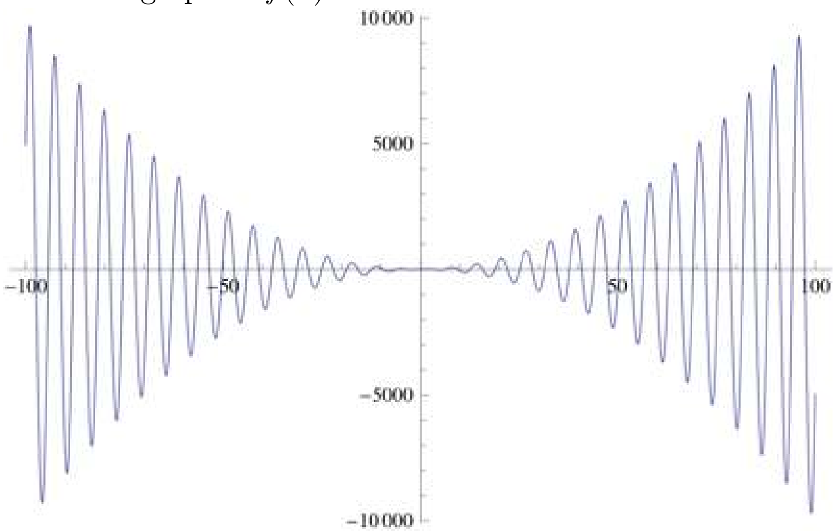
$$\lim_{x\to 1^-} f(x) = 2,$$

so $\lim_{x\to 1} \quad 10^{f(x)} = 10^2 = 100$.

2. Define

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

a. Draw the graph of $f(x)$.



b. Where is $f(x)$ continuous? Use the limit definition of continuity to explain your answer.

$\sin(x)$ is continuous everywhere, and $\frac{1}{x}$ is continuous everywhere except at 0, so $f(x)$ is continuous everywhere, except possibly at 0. By the squeeze theorem, using

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

we see that $\lim_{x\to 0} f(x) = 0$. Since $f(0) = 0$, the function is equal to the value of its limit, so by the definition of continuity, $f(x)$ is continuous at 0 as well.

3. Use the limit definition of the derivative to find the derivatives of the following functions. (Note: No credit will be given for using the power rule, it's not on this exam!)

a. $f(x) = 2x^2 + x$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)^2 + (x+h) - (2x^2+x)}{h} \\
&= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 + (x+h) - (2x^2+x)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{x} + h - \cancel{2x^2} + \cancel{x})}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 + h}{h} = \lim_{h\to 0} 4x + 2h + 1 = \boxed{\text{4x+1}}
\end{aligned}
$$

b. $f(x) = \dfrac{1}{x+2}$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+2-(x+2+h)}{(x+2)(x+2+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{(x+2)(x+2+h)}}{h} \\
&= \lim_{h\to 0} \frac{-1}{(x+2)(x+2+h)} = \boxed{-\frac{1}{(x+2)^2}}
\end{aligned}
$$

c. $f(x) = \sqrt{x}$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
&= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}
\end{aligned}
$$

4. State the precise $(\epsilon - \delta)$ definition of the limit, and use it to prove that

$$\lim_{x \to 3} -2x + 1 = -5.$$

The limit of $-2x + 1$ as x approaches 3 equals -5 if, for every $\epsilon > 0$, there exists a $\delta > 0$, such that if $|x - 3| < \delta$, then $|(-2x + 1) - (-5)| < \epsilon$.

Take $\delta = \epsilon/2$. Then if $|x - 3| < \delta = \epsilon/2$, we have that $2|x - 3| < \epsilon$, so

$$|-2x + 6| = |-2x + 1 - (-5)| < \epsilon.$$

5. Prove that there exists a value of x for which the function $f(x) = -100cos(\pi x) + 10^x$ is equal to zero.
$-100cos(\pi x) + 10^x$ is continuous everywhere. Since $f(0) = -100 + 1 = -99$ and $f(1) = 100 + 10 = 110$, there exists a value of x between 0 and 1 where $f(x) = 0$ by the intermediate value theorem.

6. According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the Pressure P and the volume V is a constant. Suppose that for a certain gas $PV = 1000$ in $\times$ pounds. Where pressure is measured in pounds per square inch, and volume is measured in cubic inches.

   a Find the average rate of change in P as V increases from 200 in^3 to 250 in^3.
   When V is 200 in^3, P is 5 $pounds/in^2$. When V is 250 in^3, P is 4 $pounds/in^2$, so the average rate of change is $\frac{\Delta P}{\Delta V} = \frac{4-5}{250-200} = \frac{-1}{50}$

   b Express V as a function of P, and find a function for the instantaneous rate of change of V with respect to P.

$$V(P) = \frac{1000}{P}.$$

$$\begin{aligned} V'(P) &= \lim_{h\to 0} \frac{\frac{1000}{P+h} - \frac{1000}{P}}{h} = 1000 \lim_{h\to 0} \frac{\frac{P-(P+h)}{P(P+h)}}{h} \\ &= 1000 \lim_{h\to 0} \frac{\frac{-h)}{P(P+h)}}{h} = 1000 \lim_{h\to 0} \frac{-1)}{P(P+h)} = \boxed{-\frac{1000}{P^2}} \end{aligned}$$

   c Find a tangent line to the function when P=50 pounds/in^2.
   $V'(50) = -\frac{1000}{2500} = -0.4$.
   So the tangent line is $V = -0.4P + c$ for some c. Plugging in the point P=50, V=20, we find $c = 40$. So the tangent line is $V = 0.4P + 40$.

7. Find all vertical and horizontal asymptotes of the function

$$f(x) = \frac{3x+8}{x+3} + \tan^{-1}(x)$$

$\frac{3x+8}{x+3}$ has a vertical asymptote at -3, so $f(x)$ does as well, this is the only vertical asymptote. To find the horizontal asymptotes, we take

$$\lim_{x\to\infty} \frac{3x+8}{x+3} + \tan^{-1}(x) = \lim_{x\to\infty} \frac{3+8/x}{1+3/x} + \lim_{x\to\infty} \tan^{-1}(x) = 3 + \pi/2$$

and

$$\lim_{x\to-\infty} \frac{3x+8}{x+3} + \tan^{-1}(x) = \lim_{x\to-\infty} \frac{3+8/x}{1+3/x} + \lim_{x\to-\infty} \tan^{-1}(x) = 3 - \pi/2$$

so the horizontal asymptotes are $3+\pi/2$ and $3-\pi/2$.

8. Draw the derivative of the following function.