# PROOF WRITING TIPS FOR ABSTRACT ALGEBRA

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## Introduction

Writing proofs in abstract algebra can feel overwhelming at first. Unlike computational problems where you plug in numbers, proofs require you to think logically and communicate your reasoning clearly. This guide provides practical strategies to help you write clear, correct proofs in this course.

## 1. General Proof Writing Strategy

1.1. **The Four-Step Process. 1. Understand what you're proving**

- Read the statement carefully
- Identify hypotheses (what you're given)
- Identify the conclusion (what you need to show)
- Ask: "What does this really mean?"

**2. Plan your approach**

- Direct proof? Contradiction? Contrapositive?
- What definitions will you need?
- What theorems or properties might help?

**3. Write a clear, logical argument**

- Start with what you know
- Proceed step-by-step using definitions and known results
- Each step should follow logically from previous steps

**4. Review and polish**

- Does each step make sense?
- Did you use all your hypotheses?
- Is your conclusion clearly stated?

## 2. Essential Proof Techniques for Abstract Algebra

2.1. **Using Definitions Effectively. Golden Rule:** When you see a mathematical term, immediately think of its definition.

**Example 1.** To prove "$H$ is a subgroup of $G$," you must show:

(1) $H$ is non-empty
(2) $H$ is closed under the group operation
(3) $H$ is closed under taking inverses

Don't just say "$H$ is obviously a subgroup"—verify each condition explicitly.

**Common Mistake:** Using intuition instead of definitions.

- **Wrong:** "Since $a$ divides $b$, we have $a < b$." (False! $-2$ divides 4)
- **Right:** "Since $a \mid b$, there exists an integer $k$ such that $b = ak$."

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2.2. **Proof by Cases.** When the hypothesis naturally splits into different scenarios, consider each case separately.

**Example 2.** To prove "If $n$ is an integer, then $n^2 \equiv 0$ or $1 \pmod 4$."

**Proof:** Let $n$ be an integer. By the division algorithm, $n = 4q + r$ where $r \in \{0, 1, 2, 3\}$.
**Case 1:** $n = 4q$. Then $n^2 = 16q^2 = 4(4q^2) \equiv 0 \pmod 4$.
**Case 2:** $n = 4q + 1$. Then $n^2 = 16q^2 + 8q + 1 = 4(4q^2 + 2q) + 1 \equiv 1 \pmod 4$.
**Case 3:** $n = 4q + 2$. Then $n^2 = 16q^2 + 16q + 4 = 4(4q^2 + 4q + 1) \equiv 0 \pmod 4$.
**Case 4:** $n = 4q + 3$. Then $n^2 = 16q^2 + 24q + 9 = 4(4q^2 + 6q + 2) + 1 \equiv 1 \pmod 4$.
In all cases, $n^2 \equiv 0$ or $1 \pmod 4$. □

2.3. **Proof by Contradiction.** Assume the opposite of what you want to prove and derive a contradiction.

**When to use:** Statements with "no," "not," or "impossible."

**Example 3. Claim:** There is no largest prime number.

**Proof:** Suppose, for the sake of contradiction, that there is a largest prime $p$. Consider the number $N = 2 \cdot 3 \cdot 5 \cdots p + 1$ (the product of all primes up to $p$, plus 1).

Since $N > p$, we know $N$ is not prime. So $N$ must have a prime divisor $q$. But $q$ cannot be any of $2, 3, 5, \ldots, p$ because dividing $N$ by any of these leaves remainder 1.

Therefore $q > p$, contradicting our assumption that $p$ is the largest prime. □

## 3. Common Mistakes and How to Avoid Them

3.1. **Logical Errors. 1. Assuming what you're proving**

- **Wrong:** "To show $a \mid bc$, we note that $a \mid bc$..."
- **Right:** "To show $a \mid bc$, we need to find an integer $k$ such that $bc = ak$..."

**2. Confusing "if" and "only if"**

- "$A$ if $B$" means "$B \Rightarrow A$"
- "$A$ only if $B$" means "$A \Rightarrow B$"
- "$A$ if and only if $B$" requires proving both directions

**3. Invalid contrapositive**

- To prove "$A \Rightarrow B$," the contrapositive is "$\neg B \Rightarrow \neg A$"
- **Wrong:** The converse "$B \Rightarrow A$" is NOT equivalent

3.2. **Presentation Issues. 1. Unclear pronouns**

- **Wrong:** "Since it divides it, we have..." (What divides what?)
- **Right:** "Since $d$ divides $n$, we have..."

**2. Missing quantifiers**

- **Wrong:** "Let $x$ be positive." (Which $x$? Any $x$? A specific $x$?)
- **Right:** "Let $x$ be any positive real number." or "Let $x$ be the positive solution to..."

**3. Insufficient justification**

- **Wrong:** "Clearly, $x = y$." (If it's clear, why mention it? If it's not clear, prove it!)
- **Right:** "By the definition of group isomorphism, $\phi(xy) = \phi(x)\phi(y)$."

## 4. LaTeX Tips for Mathematical Writing

Since you'll be writing proofs in LaTeX for this course, here are essential commands:

### 4.1. Proof Environments.

```
\begin{proof}
Your proof here.
\end{proof}
```

### 4.2. Mathematical Symbols.

- Divides: `\mid` produces $\mid$
- Does not divide: `\nmid` produces $\nmid$
- Implies: `\Rightarrow` produces $\Rightarrow$
- If and only if: `\Leftrightarrow` produces $\Leftrightarrow$
- For all: `\forall` produces $\forall$
- There exists: `\exists` produces $\exists$
- End of proof: `\qed` or just end the proof environment

### 4.3. Text in Math Mode.

- `$x \text{ is odd}$` produces $x \text{ is odd}$ (note the spaces!)
- `$\text{gcd}(a,b) = 1$` produces $\text{gcd}(a,b) = 1$

## 5. Practice Problems

Try writing proofs for these statements. Focus on clear logic and good presentation.

(1) If $a$ and $b$ are odd integers, then $a+b$ is even.
(2) If $n^2$ is even, then $n$ is even.
(3) The sum of any three consecutive integers is divisible by 3.
(4) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

## Final Advice

- **Practice regularly.** Proof writing is a skill that improves with practice.
- **Read your proofs aloud.** If it sounds awkward, it probably needs revision.
- **Get feedback.** Share your proofs with classmates or bring them to office hours.
- **Study good examples.** Pay attention to how proofs are written in your textbook.
- **Be patient.** Even experienced mathematicians revise their proofs multiple times.

Remember: A good proof convinces a skeptical but reasonable reader. If you can explain your reasoning clearly to a classmate, you're on the right track!