

MATH 267 Fall 2026 - Class Notes

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Summary: We started by going over Quiz 4. Then, we did example problems on set collections and induction. We have our first test next Wednesday.

Notes:

- Notation: UNION:

$$\bigcup_{i=1}^k A_i = A_1 \cup A_2 \cup \dots \cup A_k = \{x \in X : x \in A_i \text{ for some } 1 \leq i \leq k\}$$

INTERSECTION:

$$\bigcap_{i=1}^k A_i = \{x \in X : x \in A_i \text{ for all } 1 \leq i \leq k\}$$

- Union: “for some”
- Intersection: “for all”
- What is:

$$\bigcup_{i=1}^{\infty} A_i$$

Means every positive integer is contained in at least one of these:

$$\bigcup_{n=1}^{\infty} A_n = \{1, 2, 3, \dots\} = \mathbb{Z}^+$$

- What is:

$$\bigcap_{i=1}^{\infty} A_i$$

It means every positive integer n is NOT an element of A_{n+1} , so the intersection is empty.

- Example of index set:

$$\bigcup_{r \in [2,3]} A_r$$

(A_r) means one set for every real number between 2 and 3. $[2,3]$ is the index set. One index for each real number.

- EXAMPLE: Suppose $A_n = (\frac{-1}{n}, 1 + \frac{1}{n})$ for each $1 \leq n \in \mathbb{N}$

$$A_1 = (-1, 2)$$

$$A_2 = (\frac{-1}{2}, \frac{3}{2})$$

$$A_3 = (\frac{-1}{3}, \frac{4}{3})$$

- Prove that

$$\bigcap_{i=1}^{\infty} A_i = [0, 1]$$

- includes 0 and 1, “closed interval”.

To show equality of these sets, we have to show

$$\bigcap_{i=1}^{\infty} A_i \subseteq [0, 1]$$

AND

$$\bigcap_{i=1}^{\infty} A_i \supseteq [0, 1]$$

$\supseteq$: Show that

$$[0, 1] \subseteq \bigcap_{i=1}^{\infty} A_i$$

Let $x \in [0, 1]$, then $0 \leq x$ and $x \leq 1$. So for every integer n , $\frac{-1}{n} \leq 0 \leq x$ and $x \leq 1 < 1 + \frac{1}{n}$
So, $x \in A_n$ since $x \in A_n$ for every n

$$x \in \bigcap_{i=1}^{\infty} A_i$$

$\subseteq$: Show that

$$\bigcap_{i=1}^{\infty} A_i \subseteq [0, 1]$$

We'll prove the contrapositive. **Goal: If $x \notin [0, 1]$ then $x \notin \bigcap_{i=1}^{\infty} A_i$**

Definitions:

Definition 1. The index set is a collection of potential subscripts of the set. It can be any collection of objects, not just integers.

Theorems and Proofs:

Theorem 1. Suppose $\{A_i\}_{i=1}^{\infty}$ is an infinite collection of sets such that for all $i \geq 1$, $A_i \subseteq A_{i+1}$. Then

$$\bigcup_{i=1}^N A_i = A_N$$

For all $N \geq 1$.

Proof. Proof by induction.

Base case: $N = 1$,

$$\bigcup_{i=1}^1 A_i = A_1$$

$$\bigcup_{i=1}^1 A_i$$

is the union of the only set A_1 . This is equal to A_1 .

Induction step: Suppose that

$$\bigcup_{i=1}^N A_i = A_N$$

Then,

$$\bigcup_{i=1}^{N+1} A_i = (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_N) \cup A_{N+1} = \left(\bigcup_{i=1}^N A_i\right) \cup A_{N+1} = A_N \cup A_{N+1}$$

We need to show that $A_N \cup A_{N+1} \subseteq A_{N+1}$ and $A_N \cup A_{N+1} \supseteq A_{N+1}$. Now, $A_{N+1} \subseteq A_N \cup A_{N+1}$ by the definition of union - $A \subseteq B \cup A$ is true for all sets A and B. Since $A_i \subseteq A_{i+1}$ for all i, we have in particular that $A_N \subseteq A_{N+1}$. (Fact: If $B \subseteq A$, then $A \cup B \subseteq A$). So, $A_N \cup A_{N+1} \subseteq A_{N+1}$. Thus, $\bigcup_{i=1}^{N+1} A_i = A_N \cup A_{N+1} = A_{N+1}$ $\square$

Proof. Suppose $x \notin [0, 1]$. Then either $x < 0$ or $x > 1$.

Case 1: Suppose $x < 0$, choose $n > \frac{-1}{x}$, so $\frac{1}{n} < -x$. So, $x < \frac{-1}{n}$. Then $x \notin A_n$, then $x \notin \bigcap_{i=1}^{\infty} A_i$

Case 2: Suppose $x > 1$. Choose n so that $1 + \frac{1}{n} < x$. Then $x \notin A_n$. So, $x \notin \bigcap_{i=1}^{\infty} A_i$. $\square$

Questions raised in class: No questions were raised that were left unsettled.