MATH 267 Fall 2026 - Class Notes

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Summary: Today's class reviewed proof by contrapositive and introduced mathematical induction. We discussed the structure of an induction proof, including the base case and inductive step, and used induction to prove formulas involving sums, inequalities, and divisibility. We also discussed Fermat numbers as an example of why observing a pattern is not enough to prove that the pattern always holds.

Notes:

Proof by Contrapositive

Suppose we want to prove the statement:

If n^2 is even, then n is even.

Instead of proving the statement directly, we can prove its contrapositive:

If n is odd, then n^2 is odd.

Suppose n is odd. Then there exists an integer k such that

$$n = 2k + 1.$$

Therefore,

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since $2k^2 + 2k$ is an integer, n^2 is odd. Thus, by contrapositive, if n^2 is even, then n is even.

Fermat Numbers and Conjectures

Fermat considered numbers of the form

$$F_k = 2^{2^k} + 1.$$

For the first few values of k,

$$F_0 = 2^1 + 1 = 3,$$
$$F_1 = 2^2 + 1 = 5,$$
$$F_2 = 2^4 + 1 = 17,$$
$$F_3 = 2^8 + 1 = 257,$$

and

$$F_4 = 2^{16} + 1 = 65537.$$

All of these numbers are prime. Based on this pattern, Fermat conjectured that

$$2^{2^k} + 1$$

is prime for every nonnegative integer k.

However, about a century later Euler showed that

$$F_5 = 2^{32} + 1$$

is divisible by 641. Therefore, F_5 is not prime.

This is an example showing that checking several cases does not prove that a statement is true in general.

Idea of Mathematical Induction

The idea behind induction is to establish a pattern for all positive integers.

Suppose $P(n)$ is a mathematical statement depending on n. To prove

$$P(n)$$

for every integer $n \geq 1$, an induction proof has two main steps:

1. **Base Case:** Prove $P(1)$.
2. **Inductive Step:** Prove
$$P(n) \to P(n+1).$$

In the inductive step, we assume that $P(n)$ is true. This assumption is called the **inductive hypothesis**. We then use this assumption to prove that $P(n+1)$ is true.

The process can be compared to an infinite chain of dominoes. The base case knocks over the first domino, and the inductive step shows that whenever one domino falls, the next one must fall.

Finding a Formula for the Sum of the First n Even Integers

Consider

$$f(n) = 2 + 4 + 6 + \cdots + 2n.$$

We first calculate several small cases:

$$f(1) = 2,$$

$$f(2) = 2 + 4 = 6,$$

$$f(3) = 2 + 4 + 6 = 12,$$
$$f(4) = 2 + 4 + 6 + 8 = 20,$$
$$f(5) = 2 + 4 + 6 + 8 + 10 = 30,$$

and

$$f(6) = 2 + 4 + 6 + 8 + 10 + 12 = 42.$$

Dividing each answer by n gives

$$\frac{f(1)}{1} = 2, \qquad \frac{f(2)}{2} = 3, \qquad \frac{f(3)}{3} = 4,$$
$$\frac{f(4)}{4} = 5, \qquad \frac{f(5)}{5} = 6, \qquad \frac{f(6)}{6} = 7.$$

This suggests the conjecture

$$\frac{f(n)}{n} = n + 1.$$

Equivalently,

$$f(n) = n(n+1).$$

We can prove this formula using induction.

Base Case:

For $n = 1$,

$$f(1) = 2.$$

The proposed formula gives

$$1(1+1) = 2.$$

Therefore, the formula is true for $n = 1$.

Inductive Step:

Assume

$$f(n) = 2 + 4 + \cdots + 2n = n(n+1).$$

This is our inductive hypothesis. We want to prove

$$f(n+1) = (n+1)(n+2).$$

Now,

$$\begin{aligned} f(n+1) &= 2 + 4 + \cdots + 2n + 2(n+1) \\ &= f(n) + 2(n+1) \\ &= n(n+1) + 2(n+1) \\ &= (n+1)(n+2). \end{aligned}$$

Therefore, the formula holds for $n + 1$. Hence, by induction,

$$\boxed{2 + 4 + 6 + \cdots + 2n = n(n+1)}$$

for every integer $n \geq 1$.

Example: An Inequality

Prove using induction that

$$n < 2^n$$

for every integer $n \geq 1$.

Base Case:

Let $n = 1$. Then

$$1 < 2^1 = 2.$$

Thus the statement is true for $n = 1$.

Inductive Step:

Suppose

$$n < 2^n.$$

We want to prove

$$n + 1 < 2^{n+1}.$$

Since $n \geq 1$,

$$1 \leq 2^n.$$

Using the inductive hypothesis,

$$n + 1 < 2^n + 2^n.$$

Therefore,

$$n + 1 < 2(2^n) = 2^{n+1}.$$

Hence,

$$\boxed{n < 2^n}$$

for every integer $n \geq 1$.

Exercise 1: Arithmetic Sum

Prove using induction that

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}$$

for every integer $n \geq 1$.

Base Case:

For $n = 1$, the left-hand side is

$$1.$$

The right-hand side is

$$\frac{1(3(1) - 1)}{2} = \frac{2}{2} = 1.$$

Thus $P(1)$ is true.

Inductive Step:
Assume

$$1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}.$$

We want to prove

$$1+4+\cdots+(3n-2)+(3(n+1)-2)=\frac{(n+1)(3(n+1)-1)}{2}.$$

Using the inductive hypothesis,

$$\begin{aligned}
1+4+\cdots+(3n-2)+(3(n+1)-2) &= \frac{n(3n-1)}{2}+3(n+1)-2 \\
&= \frac{n(3n-1)+6n+2}{2} \\
&= \frac{3n^2+5n+2}{2} \\
&= \frac{(n+1)(3n+2)}{2} \\
&= \frac{(n+1)(3(n+1)-1)}{2}.
\end{aligned}$$

Therefore, the formula holds for $n+1$, so

$$\boxed{1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}}$$

for all $n \geq 1$.

Exercise 2: Divisibility

Prove using induction that

$$3 \mid (n^3+2n)$$

for every integer $n \geq 1$.

Base Case:
For $n=1$,

$$1^3+2(1)=3.$$

Since

$$3 \mid 3,$$

the statement is true for $n=1$.

Inductive Step:
Assume

$$3 \mid (n^3+2n).$$

Then there exists an integer k such that

$$n^3 + 2n = 3k.$$

We want to show

$$3 \mid \left((n+1)^3 + 2(n+1)\right).$$

Expanding,

$$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 \\ &= n^3 + 3n^2 + 5n + 3 \\ &= (n^3 + 2n) + (3n^2 + 3n + 3).\end{aligned}$$

Using the inductive hypothesis,

$$n^3 + 2n = 3k.$$

Therefore,

$$\begin{aligned}(n+1)^3 + 2(n+1) &= 3k + 3(n^2 + n + 1) \\ &= 3(k + n^2 + n + 1).\end{aligned}$$

Since $k + n^2 + n + 1$ is an integer,

$$3 \mid \left((n+1)^3 + 2(n+1)\right).$$

Thus,

$$\boxed{3 \mid (n^3 + 2n)}$$

for every integer $n \geq 1$.

Exercise 3

Another induction exercise considered in class was the inequality

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}, \qquad n \geq 1.$$

The goal is to prove this inequality by establishing the base case and then assuming the inequality for n in order to prove the corresponding inequality for $n+1$.

Definitions:

Definition 1. *A* ***conjecture*** *is a mathematical statement that is believed to be true based on observed evidence or patterns, but has not yet been proved.*

Definition 2. *A* ***counterexample*** *is an example that shows that a general statement or conjecture is false.*

Definition 3. *In a proof by mathematical induction, the* ***base case*** *is the first value for which the statement $P(n)$ is verified.*

Definition 4. *The **inductive hypothesis** is the assumption that $P(n)$ is true for an arbitrary value of n during the inductive step.*

Definition 5. *The **inductive step** consists of proving*

$$P(n) \to P(n+1).$$

That is, assuming $P(n)$ is true, we prove that $P(n+1)$ is true.

<u>Theorems and Proofs:</u>

Theorem 1. *If n is an integer and n^2 is even, then n is even.*

Proof. We prove the contrapositive. Suppose n is odd. Then

$$n = 2k + 1$$

for some $k \in \mathbb{Z}$. Therefore,

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Thus n^2 is odd. Therefore, by contrapositive, if n^2 is even, then n is even. □

Theorem 2. *For every integer $n \geq 1$,*

$$2 + 4 + 6 + \cdots + 2n = n(n+1).$$

Proof. For the base case $n = 1$,

$$2 = 1(1+1).$$

Assume

$$2 + 4 + \cdots + 2n = n(n+1).$$

Then

$$\begin{aligned} 2 + 4 + \cdots + 2n + 2(n+1) &= n(n+1) + 2(n+1) \\ &= (n+1)(n+2). \end{aligned}$$

Thus the result follows by mathematical induction. □

Theorem 3. *For every integer $n \geq 1$,*

$$n < 2^n.$$

Proof. For $n = 1$,

$$1 < 2.$$

Assume $n < 2^n$. Since $1 \leq 2^n$,

$$n + 1 < 2^n + 2^n = 2^{n+1}.$$

Therefore the result follows by induction. □

Questions raised in class:

- How can we recognize when mathematical induction is the appropriate proof technique?
- Why is checking many examples not enough to prove a statement for all integers?
- How do we determine the correct expression for $P(n+1)$ when working with sums?
- How can the inductive hypothesis be used effectively when proving divisibility statements?
- Can the inequality
$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$
be completed using the same induction structure as the other examples?