MATH 267 Fall 2026 - Class Notes

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Summary: Today's class was about sets. We reviewed the notation nuances between $\in$ ("is an element of") and $\subseteq$ ("is a subset of"), defined subsets, set equality via "double inclusion," and proper subsets. We introduced the *element method* for proving that one set is a subset of another, discussed the empty set and vacuous truth, and used counterexamples to show something is *not* a subset. We worked several full proofs: a subset proof for two sets defined by inequalities, a double-inclusion proof that $\{x \in \mathbb{R} : |x-3| < 2\} = \{x \in \mathbb{R} : 1 < x < 5\}$, a proof that $\{x \in \mathbb{Q} : x^2 = 2\} = \emptyset$, and the classic proof by contradiction that $\sqrt{2}$ is irrational.

Notes: Warm-up review (notation nuances). $\in$ means "is an element of"; $\subseteq$ means "is a subset of."

- $1 \in \{1,2,3\}$ - True
- $\{1\} \subseteq \{1,2,3\}$ -True
- $\{1\} \in \{1,2,3\}$ - False
- $\{1\} \in \{\{1\},\{2\}\}$ - True
- $1 \ \subseteq \{1,2,3\}$ - Doesn't make sense

Note (course Logistics): do not copy and paste text into Overleaf. You should "show" your thought process in Overleaf, which logs your typing/deleting.

Subsets. We use the subset relation to define what it means for two sets to be equal ("double inclusion"). Proper containment $\subset$ is sort of like less-than signs: for real numbers x and y,

$$x < y \iff x \leq y \text{ and } x \neq y.$$

You have to be careful here: sometimes people use $\subset$ when they mean $\subseteq$. Sometimes, to emphasize that $S \neq T$, we write $S \subsetneq T$. $\subsetneq$ and $\subset$ mean the same thing, but the first emphasizes the fact that we don't allow equality. *Example.* Let $S = \{1,2,3\}$ and $T = \{1,2,3,4\}$. Then $S \subset T$, $S \subseteq T$ ($\checkmark$), or $S \subsetneq T$, and $S \neq T$. True or False, with $S = \{1,2,3,\}$, $T = \{1,2,3,4\}$:

True	**False**
$S \subset T$	$S \in T$
$S \subseteq T$	$T \subset S$
$S \neq T$	$S = T$
$S \subsetneq T$	$S \supset T$
$S \notin T$	
$T \notin S$	

Also note the equivalences

$$S \subset T \iff T \supset S, \qquad S \subseteq T \iff T \supseteq S.$$

Sometimes it's not so obvious if something is a subset or not; so, we have to find a way to prove it.
The element method. In order to prove that $S \subseteq T$ we use the *element method*.

- *Idea:* Let x be an arbitrary element of S. Prove that x is also an element of T.

The empty set and vacuous truth. $\emptyset$ is the empty set, $\emptyset = \{\}$ (the set with no elements), and $\emptyset \subseteq S$ for any set S. Here $\emptyset \subseteq S$ means

$$\forall x\, (x \in \emptyset \to x \in S),$$

which is *vacuously true*, since "$x \in \emptyset$" never happens; there are no elements in the empty set. In general, $P \to Q$ is true if P never happens ("vacuously true"). The empty set is always contained inside any set for this reason.
Set equality via the element method. In order to prove that two sets are equal using the element method, we need to use the element method twice: once to show $S \subseteq T$ and then again for $T \subseteq S$.
Worked example. Let

$$A = \{x \in \mathbb{R} : (x-2)^2 < 1\}, \qquad B = \{x \in \mathbb{R} : x^2 - 4x < 5\}.$$

Goal: prove that $A \subseteq B$, and then ask whether $A = B$. Writing tips from this example:

- Your proof should read like complete sentences, which include periods.
- "$x^2 - 4x < -3 < 5$" isn't wrong, but without "So" it's not a complete sentence.
- "Thus" is often used when you're getting close to the end.
- Important to make sure letters for sets of numbers use the double bar (blackboard bold), e.g. $\mathbb{Q}$, via LaTeX (`\mathbb{Q}`).
- To prove something *is* a subset $\to$ element method. To prove something is *not* a subset $\to$ counterexample, to show $\exists$ an element (of the first set that is not in the second).
- A subset statement means that if you pick something in the first set, you need to find it in the 2nd set.
- We can give set names (like U and V below) to reduce overly repeating ourselves and writing sets again and again.
- We're allowed to do anything we want to a chain of inequalities like $1 < x < 5$, as long as we do the same to each term.

Definitions:

Definition 1. *$S \subseteq T$ if $\forall x \in S$, $x \in T$.*

Definition 2. *$S = T$ if $S \subseteq T$ and $T \subseteq S$. ("Double inclusion")*

Definition 3. *We say $S \subset T$ if $S \subseteq T$ and $S \neq T$.*

Definition 4. *$\emptyset$ is the empty set: $\emptyset = \{\}$, the set with no elements. For any set S, $\emptyset \subseteq S$.*

Theorems and Proofs:

Theorem 1. *Let $A = \{x \in \mathbb{R} : (x-2)^2 < 1\}$ and $B = \{x \in \mathbb{R} : x^2 - 4x < 5\}$. Then $A \subseteq B$.*

Proof. Prove that $A \subseteq B$: this means $\forall x \in A \to x \in B$. Use the element method (direct proof). Suppose that $x \in A$. Then $x \in \mathbb{R}$ and $(x-2)^2 < 1$. This means $(x^2 - 4x + 4) < 1$. So, $x^2 - 4x < -3 < 5$. Thus, $x^2 - 4x < 5$, so $x \in B$. Therefore, $A \subseteq B$. □

Theorem 2. *With A, B as in the previous theorem: $B \not\subseteq A$, and thus $A \neq B$.*

Proof. Is $A = B$? If we wanted to prove this we would need to prove that $B \subseteq A$. No: $A \neq B$. We need to show $B \not\subseteq A$.

Take $x = 3$. $3 \in B$ since $3^2 - 4(3) = -3 < 5$. But $3 \notin A$ since $(3-2)^2 = 1^2 = 1$, which is not less than 1. Since $3 \in B$ and $3 \notin A$, $B \not\subseteq A$. Thus, $B \neq A$. □

Note: We could find that $A = (1, 3)$ (open interval of $\mathbb{R}$) and $B = (-1, 5)$ (open interval).

Theorem 3. *$\{x \in \mathbb{R} : |x-3| < 2\}$ is equal to $\{x \in \mathbb{R} : 1 < x < 5\}$.*

Proof. (Using double inclusion.) Let $U = \{x \in \mathbb{R} : |x-3| < 2\}$ and $V = \{x \in \mathbb{R} : 1 < x < 5\}$. We will show $U \subseteq V$ and $V \subseteq U$.

First, we show $U \subseteq V$. Suppose $x \in U$; then $x \in \mathbb{R}$ and $|x-3| < 2$. Suppose $x - 3 \geq 0$; then $x - 3 < 2$. Now suppose that $x - 3 < 0$; then $x - 3 > -2$. In either case $-2 < x - 3 < 2$. Adding 3 to each term, we find that $1 < x < 5$ and thus $x \in V$.

Now, we show $V \subseteq U$. Suppose $x \in V$; then $1 < x < 5$. By subtracting 3 from each term, we find that

$$1 - 3 < x - 3 < 5 - 3.$$

So, $-2 < x - 3 < 2$. Thus, $|x-3| < 2$ by the definition of absolute. Therefore, $x \in U$. Since $U \subseteq V$ and $V \subseteq U$ we have $U = V$. □

Theorem 4. *$\{x \in \mathbb{Q} : x^2 = 2\} = \emptyset$.*

Proof. $\emptyset \subseteq \{x \in \mathbb{Q} : x^2 = 2\}$ since the empty set is a subset of every set. Since if $x \in \mathbb{Q}$ and $x^2 = 2$ then $x = \pm\sqrt{2}$. But $\sqrt{2}$ is irrational (i.e. $\sqrt{2} \notin \mathbb{Q}$). There are no elements of $\{x \in \mathbb{Q} : x^2 = 2\}$, and so the direction $\{x \in \mathbb{Q} : x^2 = 2\} \subseteq \emptyset$ is vacuously true.

Thus, $\{x \in \mathbb{Q} : x^2 = 2\} = \emptyset$. □

Theorem 5. *$\sqrt{2}$ is irrational.*

Proof. Suppose for contradiction that $\sqrt{2}$ is rational. Then, $\sqrt{2} = \frac{a}{b}$ where a and b are integers, $b \neq 0$. We can assume that a and b don't have any common factors (i.e. $\frac{a}{b}$ is a reduced fraction).

Then $b\sqrt{2} = a$. Squaring both sides, we find that $2b^2 = a^2$. Then, a^2 is an even number. This means that a is even by a fact we proved earlier. Since a is even, $a = 2k$ for some integer k. Thus, $2b^2 = (2k)^2 = 4k^2$. Dividing through by 2 we have $b^2 = 2k^2$. So, b^2 is even and thus b is too.

Thus, a and b are both even, but this contradicts the assumption that a and b have no common factors. $\rightarrow\leftarrow$ $\square$

Questions raised in class: None