MATH 267: INTRODUCTION TO ABSTRACT MATH
LECTURE NOTES - AUGUST 24, 2026

DR. MCNEW

Reading: Chapter 1 of Gorkin & Daepp. **Notes:** §1.

Goals for today. By the end of class you should be able to:

- explain what distinguishes a *theorem* from a plausible-sounding claim;
- state the difference between an axiom, a definition, and a theorem;
- use a formal definition exactly as written, ignoring what the word means in everyday English;
- apply Pólya's four steps to an unfamiliar problem.

Plan (110 minutes). Welcome (12) · Activity 1 (20) · Rules of the game (20) · Activity 2 (10) · Divisibility (20) · Activity 3 (15) · Pólya + cipher (13).

1. Welcome and Course Mechanics (12 min)

Brief tour of the course website. Three things students must act on this week:

- **Reading reflections** are due at 11:59pm *the night before* each class with new material — so the first one is due tomorrow, August 25. Set up Overleaf today.
- **Course notes sign-up**: each student types up notes for two lectures in LaTeX.
- Bring the textbook, or the notes, to every meeting.

Say plainly what makes this course different from Calculus: *the answer is no longer a number, and the work is no longer a computation.* From here on, the object you produce is an argument, and an argument is either valid or it is not.

2. Activity 1: Which of These Are Not Theorems? (20 min)

Put the following list on the board (or hand it out). Groups of three. Tell them exactly this: nine of these eleven statements are true theorems, and two are false. Find the two, and be ready to justify your answer to the class.

(1) $\sqrt{2}$ is irrational.
(2) If n is an integer, then $\sqrt[3]{n}$ is irrational.
(3) Every integer $n \geq 2$ can be expressed as a product of primes.
(4) There are infinitely many primes.
(5) The sum of two rational numbers is a rational number.
(6) The sum of two irrational numbers is an irrational number.
(7) If $f\colon (0,\infty) \to \mathbb{R}$ is increasing and $f(x) \leq 1$ for all $x > 0$, then $y = f(x)$ has a horizontal asymptote as $x \to \infty$.
(8) If A and B are 3×3 matrices, then $\det(AB) = (\det A)(\det B)$.
(9) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$.
(10) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$.
(11) $\displaystyle\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$.

> **Activity 1.** Find the two false statements. For each one, give a specific counterexample.

Why this activity. Every statement is stated in the same confident voice, and two are simply wrong. The only defense is checking, and the cheapest check is a counterexample. This is the habit the whole course depends on.

Instructor notes. The two false statements are (2) and (6).

- (2) fails at any perfect cube: $n = 8$ gives $\sqrt[3]{8} = 2$, which is rational. Students often get distracted defending the $\sqrt{2}$ case and never test small integers. Good moment to say: *when you doubt a "for all" claim, try the smallest and stupidest inputs first.*
- (6) fails at $\sqrt{2} + (-\sqrt{2}) = 0$. Nearly every group finds this one. Push them: is there a rule about when the sum *is* irrational? (There isn't a simple one — good.)
- Expect (7) to be challenged. It is true: an increasing function bounded above by 1 converges as $x \to \infty$. Worth naming that they are already relying on completeness of $\mathbb{R}$, which we prove in October.
- Expect (11) to be challenged too, over the missing $|\cdot|$ or the domain. Kumchev's point: whether a fact "counts" as a theorem is partly context. Formally it is one, and it has a proof.

3. The Rules of the Game (20 min)

Lecture. The point is not to be exhaustive; it is to make the vocabulary precise.

- **Undefined (primitive) terms.** Every term must be defined — but the process has to start somewhere, so a few terms are left undefined by agreement. *Set* and *element* are ours.
- **Definitions.** Once an English word is given a mathematical definition, its everyday meaning becomes irrelevant. This is the rule students violate most often, and it is worth saying twice.
- **Axioms.** Facts accepted without proof, again because the process must start somewhere.
- **Theorems.** Facts deduced from the axioms by valid logic. Subtypes are organizational only: *lemma* (a stepping stone), *proposition* (a modest result), *corollary* (a quick consequence). A *conjecture* is not yet any of these.

Worth stating out loud: if two branches of mathematics adopt different axioms, each can prove things the other denies, and both are correct *within their own system.* The parallel postulate is the famous case. So "true" always means "true given these axioms."
Notation is terminology in disguise. If you meet a symbol you cannot define, you have not read the sentence — you have looked at it.

4. Activity 2: Define $\mathbb{N}$ (10 min)

Activity 2. Complete the sentence: "The set $\mathbb{N}$ of natural numbers is ________________________." Write it down before discussing with anyone.

Collect answers. The class will split over whether $0 \in \mathbb{N}$, and that is the entire point: a term they have used since childhood has no universally agreed definition. Textbooks genuinely differ.

Our convention for this course: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. The first natural number is 0. Record the rest of the standard notation now:

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}, \qquad \mathbb{Q} = \{a/b : a, b \in \mathbb{Z},\ b \neq 0\}, \qquad \mathbb{R}, \qquad \mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}.$$

5. A First Real Definition: Divisibility (20 min)

Definition 1. Let a and b be integers. We say that b **divides** a if $a = bq$ for some integer q. When b divides a we write $b \mid a$; otherwise $b \nmid a$.

Work these at the board, slowly, and insist the class justify each one *from the definition* rather than from intuition about division:

$$2 \mid 6, \qquad 3 \mid 6, \qquad 0 \mid 0, \qquad 4 \nmid 6, \qquad 0 \nmid 5, \qquad 5 \mid 0.$$

The last three are the whole lesson. Note that the definition never mentions division, only multiplication — which is exactly why $0 \mid 0$ is true ($0 = 0 \cdot q$ for any q) and $0 \nmid 5$ is false ($5 = 0 \cdot q$ has no solution). Anyone reasoning "you cannot divide by zero" is using the lay meaning instead of the definition, which is the mistake we just warned about.

Activity 3. Give three pairs a, b with $b \mid a$ and two pairs with $b \nmid a$, using only integers greater than 10. Then decide: is there an integer a with $a \mid 0$? Is there one with $0 \mid a$?

6. Activity 4: Reducible and Irreducible (15 min)

Definition 2. An integer $n > 1$ is **reducible** if $n = ab$ for integers a, b with $1 < a, b < n$. If $n > 1$ is not reducible, n is **irreducible**.

For example 6 is reducible, since $6 = 2 \cdot 3$ and $1 < 2, 3 < 6$. And 3 is irreducible: if $3 = ab$ with $1 < a, b < 3$ then $a = b = 2$, but $2 \cdot 2 \neq 3$.

Activity 4. Give three reducible integers and two irreducible ones, and explain why each works *using the definition.*

Do not call these "prime" and "composite." Say explicitly that we are reserving those words — in two weeks we will define *prime* a completely different way, and then prove the two notions agree. Students will find this pedantic today. That is fine; the payoff is Chapter 2.

7. Pólya's Four Steps, and a Cipher (13 min)

Present the four steps from Chapter 1: *understand the problem*, *devise a plan*, *carry out the plan*, *look back*.

Then run Exercise 1.1 as a live demonstration. Every letter has been shifted the same number of places down the alphabet:

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PDEO AJYKZEJC WHCKNEPDI EO YWHHAZ W YWAOWN YELDAN.
EP EO RANU AWOU PK XNAWG, NECDP?
```

Walk the four steps aloud rather than just solving it. *Understand*: the unknown is a single number n, the shift. *Plan*: attack the short words — `W`, `EO`, `EP`, `PK` — because guessing a one- or two-letter word is a much easier problem. *Carry out*: try `W` = "a". *Look back*: does the whole message read sensibly?

Finish it at home if time runs short, and make the closing point: solving it is not the same as being done. Being done means a written report, in complete English sentences, that another person can follow.

Instructor note. The shift is $n = 22$ (so $a \mapsto w$). The message reads: "THIS ENCODING ALGORITHM IS CALLED A CAESAR CIPHER. IT IS VERY EASY TO BREAK, RIGHT?" If a group finishes early, ask how they would break it *without* guessing short words — letter frequency, or simply trying all 25 shifts. Both are worth naming as plans.

8. Assigned for Wednesday

- Read Chapter 2 (Kumchev §2).
- Reading reflection due Tuesday 8/25 at 11:59pm on Overleaf.
- Finish the cipher; write the solution as a paragraph, not as a table.
- Sign up for two lecture-note dates.