MATH 267 Fall 2026 - Class Notes

11/09/2026

Scribe: Sierra Weems

Summary: This lecture introduced how a sequence converges to a limit when its terms eventually get closer to that limit than any small distance ϵ. To prove that a sequence converges, you must guess the limit, simplify the algebraic distance, find a simpler upper bound, and solve for a lower bound N using scratch work. If a sequence fails to approach a limit, it diverges, which can be proven by contradiction by assuming a limit exists and showing that specific terms break the rule.

Notes:

Question:

For any $\epsilon > 0$, can we guarantee that our sequence is eventually always closer to the limit than ϵ? If so, then the limit exists (If not, then this is either not the right limit or the sequence diverges).

From Warm-up: $x_n = \frac{4n+1}{2n-3} \to 2$

How large do we have to make N before $|x_n - 2| < 0.1$?
If $n > 45$ then $|x_n - 2| < 0.1$.

$x_n = \frac{4n+1}{2n-3}$ and $\epsilon = 0.1$ then, $\forall n > 36, |x_n - 2| < \epsilon$. (computationally)

Proof Strategy to Prove Convergence of a Sequence:

1. Guess the value of the limit L using techniques from calculus.
2. Simplify $|x_n - L|$ algebraically.
3. Find a simpler upper bound for it (replace expressions with bigger quantities).
4. Set it equal to epsilon ϵ and solve for n to get a lower bound for N.
5. Write down a proof that for all $n > N, |x_n - L| < \epsilon$.

Example 1

Prove that $x_n = \frac{5n-2}{2n+3}$ converges to $\frac{5}{2}$.
We know that $L = \frac{5}{2}$.

Proof: Fix $\epsilon > 0$ (goal: show $|x_n - \frac{5}{2}| < \epsilon$ for all $n > N$).
Set $N = \frac{5}{\epsilon}$ (figured out from scratch work $\Rightarrow$ should be done first)
Compute $|x_n - \frac{5}{2}| = |\frac{5n-2}{2n+3} - \frac{5}{2}|$
$= |\frac{2(5n-2)}{2(2n+3)} - \frac{5(2n+3)}{2(2n+3)}|$
$= |\frac{10n-4-(10n+15)}{2(2n+3)}$
$= |\frac{-19}{4n+6}| = \frac{19}{4n+6} = \frac{19}{4n}$

Now, for every $n > \frac{5}{\epsilon}, |x_n - \frac{5}{2}| < \frac{19}{4n} < \frac{19}{4(\frac{5}{\epsilon})} = \frac{19\epsilon}{20} < \epsilon$.

Scratch work:(Why did we pick $N = \frac{5}{\epsilon}$?)
Start with $|x_n - \frac{5}{2}| = |\frac{-19}{4n+6}| = \frac{19}{4n+6} < \frac{19}{4n}$
We want $\frac{19}{4n} < \epsilon \Rightarrow 19 < \epsilon \cdot 4n \Rightarrow \frac{19}{4\epsilon} < n$
$\Rightarrow$ (We want to pick something larger and simpler than $\frac{19}{4\epsilon}$)
$\Rightarrow$ Use $\frac{19}{4\epsilon} < \frac{5}{\epsilon}$ (since 20 in the numerator is a simpler number)

Example 2

$x_n = 2(-1)^n$
Prove that x_n diverges.

Proof by Contradiction:
Suppose x_n converges to some L.
Pick $\epsilon = 1$, then there exists an N so that if $n > N$ then $|x_n - L| < \epsilon = 1$.

Suppose $L \geq 0$.
Pick and odd $n < N$, then $|x_n - L| = |-2- = L| \geq |-2| = 2 > \epsilon = 1$ (Since $L \geq 0$ there is contradiction with $|x_n - L| < 1$ for all $n > N$).

Now suppose, $L < 0$.
Pick an even $n < N$, then $|x_n - L| = |2 - L| > 2$ since $L > 0$ and this again is greater than $\epsilon = 1$, a contradiction.

Definitions:

Definition 1. *We say a sequence $\{x_n\}$ converges to L if for any $\epsilon > 0$, $\exists N$ such that for all $n > N$, $|x_n - L| < \epsilon$.*

Definition 2. *If our sequence does not converge to a limit L, we say the sequence diverges.*

Theorems: No theorems covered in this lecture.
Questions raised in class: All questions were fully settled in class.