MATH 267 Fall 2026 - Class Notes

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Summary: Today we covered sequences, how they're defined, and how we can do the same things we do to sets to them. We also covered a part of the last quiz to help with our understanding.

Notes:

- First we covered part of the last quiz, specifically the second question.
  2. Let $f : A \to B$ and $g : B \to C$ be functions. Suppose f and g are both onto. Prove that $g \circ f$ is onto.
  Our goal here is to show $g \circ f$ is onto, recall that to do this we must pick an arbitrary element in the codomain and show that there exists a corresponding element in the domain such that the function value at that domain element is the arbitrary element in the codomain.
  Proof: Let $c \in C$, since g is onto $\exists b \in B$ such that $g(b) = c$, since f is onto $\exists a \in A$, such that $f(a) = b$, now $g \circ f(a) = g(f(a)) = g(b) = c$, since c is arbitrary $g \circ f$ is onto. $\square$
  Also this brought the question asked by a student about composite functions where if $f : A \to B$ and $g : C \to D$, $g \circ f$ is only defined if $B \subseteq C$.

- A sequence x_n of real numbers is a function from $f : \mathbb{N} \to \mathbb{R}$ where $x_n = f(n)$.
  Alternatively every sequence is also a set of real numbers $\{x_n : n \in \mathbb{N}\}$.
  The infimum and supremum of sequences is exactly the same as the corresponding set of real numbers.

- We went over an example in the warm up where $x_n = \frac{2n-1}{n+2}$, we found the first elements in the sequence to be $x_0 = \frac{-1}{2}, x_1 = \frac{1}{3}, x_2 = \frac{3}{4}, x_3 = 1$ which appears to be increasing. Since it appears to be increasing we can conjecture that $\frac{-1}{2}$ is the infimum (theorem 1 down below) and since as we use larger cases of n the sequence gets closer to 2 so we conjecture that the supremum is 2. We can see it's an upper bound since $\frac{2n-1}{n+2} < \frac{2n}{n+2} < \frac{2n}{n} = 2$, proving its the infimum would require showing that no smaller number is also an upper bound.

- A formula for a sequence where x_{n+1} depends on x_n(and maybe also x_{n-1}, x_{n-2} and so on) is called a recursive definition.
  If we can write $x_n = f(n)$(doesn't involve sequence terms) this is called a closed form of our sequence.
  Usually we prefer closed forms to recursive definitions. Often we use induction to prove

a recursive definition and closed form give the same definition of the sequence.
For example from the warm up today $a_1 = 1$ and $a_{n+1} = 2a_n + 1$, we can conjecture that its closed form is $f(n) = 2^n - 1$.

Definitions:

Definition 1. *A sequence is bounded above if there exists an M with $M \geq x_n \forall n$, and its bounded below if there exists m with $m \leq x_n \forall n$. It is bounded if it is bounded above and below.*

Definition 2. *The infimum exists if and only if it is bounded below, the supremum exists if and only if it is bounded above.*

Theorems and Proofs:

Theorem 1. *If x_n is increasing then the first term is the infimum(also minimum).*

Worked example. $x_n = \frac{n}{n+3}$ Prove that x_n is bounded, is strictly increasing and find and prove its infimum and supremum.

Proof. Bounded: Find $M \geq \frac{n}{n+3}, \forall n$, since $n < n + 3$, dividing both sides by $n + 3$ gives $\frac{n}{n+3} < \frac{n+3}{n+3} = 1$ so 1 is an upper bound. Since $n \geq 0, \forall n$ and $n + 3 > 0, \forall n$ then $\frac{n}{n+3} \geq 0$ so 0 is a lower bound for the sequence and since its bounded above and below its bounded.
To show x_n is strictly increasing compute $x_{n+1} - x_n$, if its positive the sequence is increasing. So $\frac{n+1}{(n+1)+3} - \frac{n}{n+3} = \frac{n+1}{n+4} - \frac{n}{n+3} = \frac{(n+1)(n+3)}{(n+4)(n+3)} - \frac{n(n+4)}{(n+3)(n+4)} = \frac{n^2+4n+3-n^2-4n}{(n+4)(n+3)} = \frac{3}{(n+3)(n+4)}$, both the numerator and the denominator are positive so $x_{n+1} - x_n$ is positive thus x_n is strictly increasing.
Since the sequence is increasing the $inf(\{x_n\}) = x_0 = 0$. Then since when we pick larger and larger values of n the sequence seems to get closer to 1 so we conjecture that the supremum of x_n is 1. To prove 1 is the supremum we must show it's an upper bound (we already did that) but we still must show its the least upper bound or that there is no smaller number that can also be an upper bound. We do this with a contradiction.
Lets suppose that $p < 1$ is also a upper bound for our sequence. Then write $x_n = \frac{n}{n+3} = \frac{n+3-3}{n+3} = \frac{n+3}{n+3} - \frac{3}{n+3} = 1 - \frac{3}{n+3}$, since $p < 1$, $1 - p > 0$, then $1 - p > \frac{3}{n+3}$, then $n > \frac{3}{1-p} - 3$, by the archemdian principal it is always possible to pick such an n that $n > \frac{3}{1-p} - 3$, then $x_n = 1 - \frac{3}{n+3} > 1 - \frac{3}{(\frac{3}{1-p}-1)+3}$(here we replaced n by something smaller than it). Then that equals $1 - \frac{3}{\frac{3}{1-p}} = 1 - (1 - p) = p$, since this x_n is larger than p $x_n > p$ and p is not an upper bound for our sequence, so 1 is the least upper bound for x_n and its supremum. □

Theorem 2. *If $f : X \to X$ is any function and $a \in X$ then there is a unique function $g : \mathbb{N} \to X$ such that $g(0) = a$ and $g(n + 1) = f(g(n)), \forall n > 0$, so $g(1) = f(a)$, $g(2) = f(f(a))$. In general $g(n) = f^{(n)}(a)$ f composed of itself n times.*
Since $g : \mathbb{N} \to X$ it gives a corresponding sequence $x_n = g(n)$.
This means we can work with recursively defined sequences and the result should always be well defined.
EX) $x_n = 3$, $x_{n+1} = 2x_n + 1$, $\forall n > 0$, gives an unique sequence (only 1 sequence satisfies this recursion).